LECTURE 13: DIRECTIONAL DERIVATIVES

1. Directional Derivatives

Let $f : \mathbb{R}^m \to \mathbb{R}$, so f is a scalar function.

Recall:

$$\frac{\partial f}{\partial x_j}(x) = \lim_{t \to 0} \frac{f(x + te_j) - f(x)}{t}$$

This is the limit in the e_j direction. But what about arbitrary directions u?

Definition: If u is a (unit) vector, then the **directional derivative** of f at x along u is

$$(D_u f)(x) = \lim_{t \to 0} \frac{f(x + tu) - f(x)}{t}$$

(There is nothing special today about u being a unit vector, it's just so that people agree on the same answer)

It turns out we can write $D_u f$ more compactly, in terms of the partial derivatives

Definition: The **gradient** of f is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

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This is just the vector of the partial derivatives

Fact:

$$D_u f(x) = (\nabla f)(x) \cdot u$$

Proof: Let $\gamma(t) = x + tu$ (segment that starts at x, in the direction of u), and let $g(t) = f(\gamma(t))$

Then by the Chain Rule,

$$\begin{aligned}
g'(t) =& f'(\gamma(t))\gamma'(t) \\
=& \begin{bmatrix} \frac{\partial f}{\partial x_j}(\gamma(t)) \end{bmatrix} \begin{bmatrix} \gamma_j'(t) \end{bmatrix} \qquad (\gamma_j' \text{ are the components of } \gamma') \\
=& \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\gamma(t))\gamma_j'(t) \\
=& (\nabla f)(\gamma(t)) \cdot \gamma'(t) \\
=& (\nabla f)(x + tu) \cdot u \quad (\text{Definition of } \gamma(t))
\end{aligned}$$

In particular, if $t = 0$ then we get

$$g'(0) = (\nabla f)(x) \cdot u$$

And therefore:

$$\begin{aligned}
(D_u f)(x) = \lim_{t \to 0} \frac{f(x + tu) - f(x)}{t} = \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} = \lim_{t \to 0} \frac{g(t) - g(0)}{t} \\
= g'(0) = (\nabla f)(x) \cdot u
\end{aligned}$$

2. Mean Value Theorem

Question: Is the Mean-Value Theorem true in $\mathbb{R}^n$? Do we have

$$f(b) - f(a) = f'(c)\,(b-a)$$

For some c in the segment between a and b?

Unfortunately the answer is **NO**.

Example: Define $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(t) = (\cos(t), \sin(t))$$

Then $f(2\pi) - f(0) = (1,0) - (1,0) = (0,0) \neq f'(c)\,(2\pi - 0)$ for any c
Since $f'(c) = (-\cos(c), \sin(c)) \neq (0,0)$

That said, we do have the following analog of the MVT, which is good enough for our purposes:

Theorem: Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and there is $M \geq 0$ such that for all x

$$\|f'(x)\| \leq M$$

Then for all a and b we have $|f(b) - f(a)| \leq M\,|b-a|$

Corollary: If $f'(x) = 0$ for all x, then f is constant.

Why? Follows from the above with $M = 0$

Proof of MVT:

STEP 1: Fix a and b and define the segment from a to b:

$$\gamma(t) = (1-t)a + tb \qquad (0 \leq t \leq 1)$$

$$\text{Let } g(t) = f(\gamma(t)) \quad \text{(Collects } f \text{ along the segment)}$$

$$\text{Then } g'(t) = f'(\gamma(t))\gamma'(t) = f'(\gamma(t))(b-a) \quad \text{(Definition of } \gamma'\text{)}$$

$$\text{Hence } \|g'(t)\| \leq \|f'(\gamma(t))(b-a)\| = \|f'(\gamma(t))\| \, |b-a| \leq M \, |b-a|$$

Notice the above is valid for all t.

Claim: There is some c with $|g(1) - g(0)| \leq \|g'(c)\|$

Then we would be done because

$$|f(b) - f(a)| = |f(\gamma(1)) - f(\gamma(0))| = |g(1) - g(0)| \overset{\text{Claim}}{\leq} \|g'(c)\| \leq M \, |b-a| \checkmark$$

STEP 2: Proof of Claim:

$$\text{Let } \phi(t) = (g(1) - g(0)) \cdot g(t) \quad \text{(Scalar function)}$$

Then by the single-variable MVT applied to ϕ there is c in $(0,1)$ with

$$\phi(1) - \phi(0) = \phi'(c) \overset{\text{DEF}}{=} (g(1) - g(0)) \cdot g'(c)$$

$$\begin{aligned}\text{But also } \phi(1) - \phi(0) &\overset{\text{DEF}}{=} (g(1) - g(0)) \cdot g(1) - (g(1) - g(0)) \cdot g(0) \\ &= (g(1) - g(0)) \cdot (g(1) - g(0)) \\ &= |g(1) - g(0)|^2\end{aligned}$$

$$\text{Hence } |g(1) - g(0)|^2 = \phi(1) - \phi(0) = (g(1) - g(0)) \cdot g'(c) \overset{\text{C-S}}{\leq} |g(1) - g(0)| \, \|g'(c)\|$$

Dividing both sides by $|g(1) - g(0)|$ we get $|g(1) - g(0)| \leq \|g'(c)\|$ $\square$