LECTURE 14: THE CHEN LU

Welcome to the most important differentiation rule of all time, the Chain Rule, or, as I like to call it, the Chen Lu!!!

1. The Chen Lu

Example 1: (Motivation)

Find the derivative of $f(t) = \sin(t^3)$

$$
\begin{aligned}
f'(t) &= \cos\left(t^3\right)\left(3t^2\right) \\
\frac{df}{dt} &= \left(\frac{df}{dx}\right)\left(\frac{dx}{dt}\right) \qquad \text{(Where } x = t^3\text{)}
\end{aligned}
$$

Here it's the same formula, except we add a $y-$term:

Example 2:

$$
\begin{cases}
\text{Find } \dfrac{df}{dt}, \text{ where} \\
f(x,y) = x^3 + y^4 \\
x = t^2 - 2 \\
y = t^3 + 1
\end{cases}
$$

In other words, find the derivative of $(t^2 - 2)^3 + (t^3 + 1)^4$. You could do it directly, but this way is more elegant and useful for applications.

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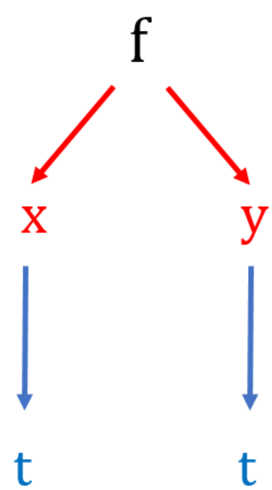
The Chen Lu:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt}\right) + \underbrace{\left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt}\right)}_{\text{New Term}}$$

You can visualize it with the following diagram:



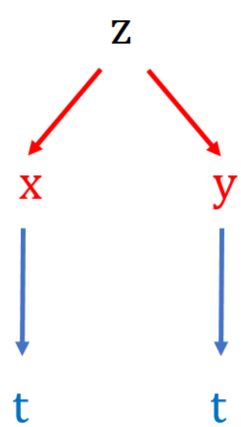
Here we get:

$$\begin{aligned}
\frac{df}{dt} &= \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt}\right) \\
&= \left(x^3 + y^4\right)_x \left(t^2 - 2\right)' + \left(x^3 + y^4\right)_y \left(t^3 + 1\right)' \\
&= \left(3x^2\right)(2t) + \left(4y^3\right)\left(3t^2\right) \\
&= 3\left(t^2 - 2\right)^2 (2t) + 4(t^3 + 1)^3 \left(3t^2\right) \qquad (\text{Use } x = t^2 - 2) \\
&= 6t\left(t^2 - 2\right)^2 + 12t^2\left(t^3 + 1\right)^3
\end{aligned}$$

Notice the pattern: First differentiate the outside function $x^3 + y^4$ and then differentiate the inside function $t^2 - 2$ (and repeat for y)

Example 3:

$$\begin{cases} \text{Find } \dfrac{dz}{dt} \\ \quad z = \ln(x^2+y^2) \\ \quad x = e^t \\ \quad y = t^2 \end{cases}$$



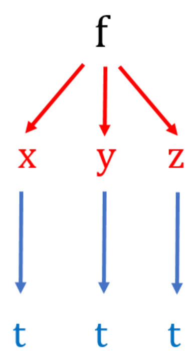
Same thing, except we use z instead of f:

$$\begin{aligned} \frac{dz}{dt} &= \left(\frac{\partial z}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{dy}{dt}\right) \\ &= \left(\ln(x^2+y^2)\right)_x \left(e^t\right)' + \left(\ln(x^2+y^2)\right)_y \left(t^2\right)' \\ &= \left(\frac{2x}{x^2+y^2}\right) e^t + \left(\frac{2y}{x^2+y^2}\right)(2t) \\ &= \left(\frac{2e^t}{(e^t)^2+(t^2)^2}\right) e^t + \left(\frac{2t^2}{(e^t)^2+(t^2)^2}\right)(2t) \\ &= \frac{2e^{2t}+4t^3}{e^{2t}+t^4} \end{aligned}$$

What if we have 3 or more variables? No problem!

Example 4:

$$\begin{cases} \dfrac{df}{dt} \text{ at } t = 0 \\ f(x,y,z) = xe^{y^2z^3} \\ x = t^2 \\ y = 1 - t^3 \\ z = 1 + 2t \end{cases}$$



Same idea as before, but this time we add a z:

$$\begin{aligned} \frac{df}{dt} &= \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt}\right) + \left(\frac{\partial f}{\partial z}\right)\left(\frac{dz}{dt}\right) \\ &= \left(e^{y^2z^3}\right)(2t) + \left(x(2y)z^3e^{y^2z^3}\right)(-3t^2) + \left(xy^2\left(3z^2\right)e^{y^2z^3}\right)(2) \end{aligned}$$

Now at $t = 0$ we have:

$$\begin{cases} x = (0)^2 = 0 \\ y = 1 - (0)^3 = 1 \\ z = 1 + 2(0) = 1 \end{cases}$$

And so: $$\frac{df}{dt} = e^1(0) + (0)(2)(1)e^1(-0) + 0(1)(3)e^1(2) = 0$$

2. Two or more variables

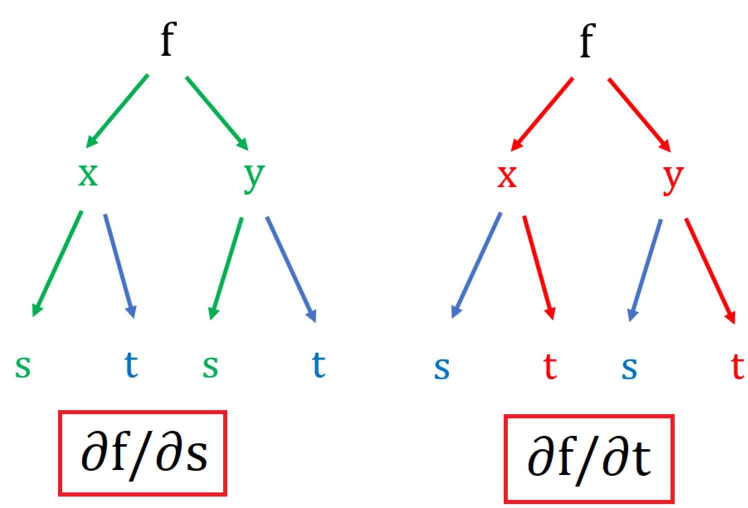
What if x and y depend on two variables t and s? Also not a problem!

Example 5:

$$\begin{cases} \text{Find } \dfrac{\partial f}{\partial s} \text{ and } \dfrac{\partial f}{\partial t} \\ f(x,y) = x\sin(y) \\ x(s,t) = s^2 t \\ y(s,t) = s^3 \end{cases}$$

In other words, we want to find the partial derivatives of $(s^2t)\sin(s^3)$

Same thing as before, except now you have to follow the arrows (kind of like a chain ☺)



The Chen Lu:

$$\frac{\partial f}{\partial s} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial s}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial s}\right)$$
$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right)$$

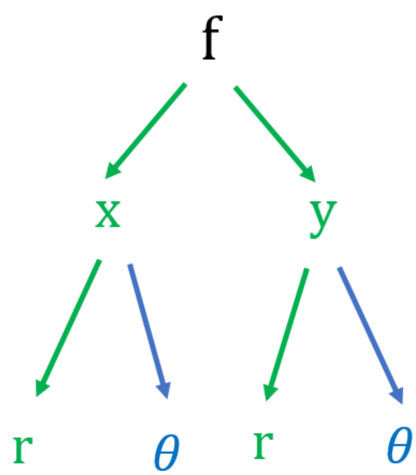
Here:

$$\begin{aligned}\frac{\partial f}{\partial s} &= \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial s}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial s}\right)\\ &= (x\sin(y))_x\left(s^2t\right)_s + (x\sin(y))_y\left(s^3\right)_s\\ &= \sin(y)\,(2st) + (x\cos(y))\left(3s^2\right)\\ &= \sin(s^3)\,(2st) + s^2t\cos(s^3)\left(3s^2\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial t} &= \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right)\\ &= (x\sin(y))_x\left(s^2t\right)_t + (x\sin(y))_y\left(s^3\right)_t\\ &= \sin(y)\left(s^2\right) + (x\cos(y))\,0\\ &= \sin(s^3)s^2\end{aligned}$$

Example 6: (Radial Derivative, extra practice)

$$\begin{cases} \text{Find } \dfrac{\partial f}{\partial r} \\ f(x,y) = \ln(x-y) \\ x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$



$$
\begin{aligned}
\frac{\partial f}{\partial r} &= \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial r}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial r}\right) \\
&= (\ln(x-y))_x \, (r\cos(\theta))_r + (\ln(x-y))_y \, (r\sin(\theta))_r \\
&= \frac{1}{x-y}\cos(\theta) - \frac{1}{x-y}\sin(\theta) \\
&= \frac{\cos(\theta)-\sin(\theta)}{x-y} \\
&= \frac{\cos(\theta)-\sin(\theta)}{r\cos(\theta)-r\sin(\theta)} \\
&= \frac{1}{r}
\end{aligned}
$$

3. Implicit Differentiation (again)

Let's revisit a problem we did in the previous section:

Example 7:

Find $\frac{\partial z}{\partial x}$ where:

$$x^3 + y^3 + z^3 = 6xyz$$

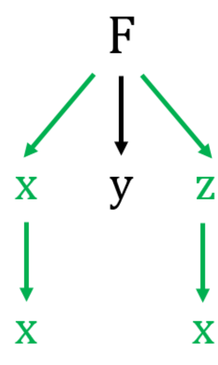
Here $z = z(x, y)$ is implicitly a function of x and y, so derivatives like $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ make sense.

We have done this example before (and it was a pain), but here we get a much easier formula:

STEP 1: Rewrite the above as:

$$\underbrace{x^3 + y^3 + z^3 - 6xyz}_{F(x,y,z)} = 0$$

Then this equation is of the form $F(x, y, z) = 0$, where both x and z depend on x.



STEP 2: Differentiate this with respect to x:

$$\begin{aligned}
(F(x,y,z))_x &= 0 \\
\frac{\partial F}{\partial x}\left(\frac{\partial x}{\partial x}\right) + \frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial x}\right) &= 0 \\
F_x + F_z\left(\frac{\partial z}{\partial x}\right) &= 0 \\
F_z\left(\frac{\partial z}{\partial x}\right) &= -F_x \\
\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z}
\end{aligned}$$

Fact:

If $F(x,y,z)=0$, then:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

Mnemonic: In $\frac{\partial z}{\partial x}$, switch the z and x and add a minus sign to get $-\frac{F_x}{F_z}$

Note: Memorize *only* if you find this useful. You can always use the method from partial derivatives section to do this.

Here, we get:

$$\frac{\partial z}{\partial x} = -\frac{\left(x^3+y^3+z^3-6xyz\right)_x}{\left(x^3+y^3+z^3-6xyz\right)_z} = -\left(\frac{3x^2-6yz}{3z^2-6xy}\right) = \frac{6yz-3x^2}{3z^2-6xy}$$

(Same answer as in the partial derivatives lecture)

Similarly:

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\left(x^3+y^3+z^3-6xyz\right)_y}{\left(x^3+y^3+z^3-6xyz\right)_z} = -\left(\frac{3y^2-6xz}{3z^2-6xy}\right) = \frac{6xz-3y^2}{3z^2-6xy}$$

Example 8:

Find $\frac{\partial z}{\partial y}$ where:

$$yz + x\ln(y) = z^2 + 1$$

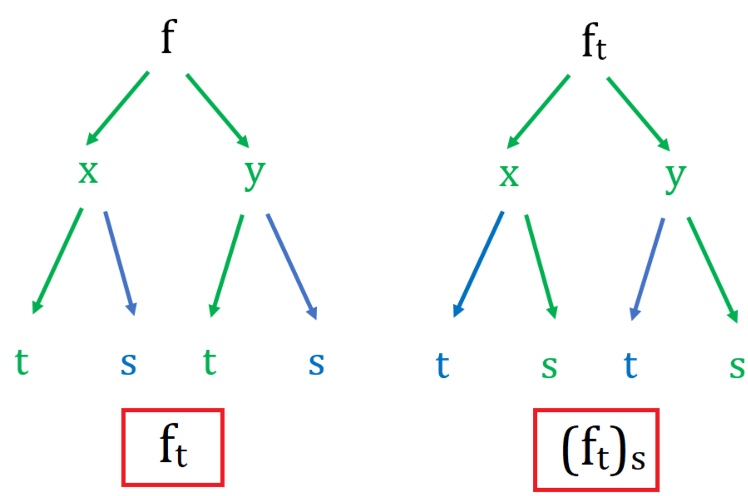
$$F(x,y,z) = yz + x\ln(y) - z^2 - 1 (= 0)$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\left(yz + x\ln(y) - z^2 - 1\right)_y}{\left(yz + x\ln(y) - z^2 - 1\right)_z} = -\left(\frac{z + \frac{x}{y}}{y - 2z}\right)$$

4. Higher Order Derivatives

Example 9: (Higher Order Derivatives)

$$\begin{cases} \text{Find } f_{ts} \\ f(x,y) = \sin(x)\cos(y) \\ x = t + s \\ y = t - s \end{cases}$$



$$\begin{aligned}
\frac{\partial f}{\partial t} &= \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right) \\
&= \left(\sin(x)\cos(y)\right)_x (t+s)_t + \left(\sin(x)\cos(y)\right)_y (t-s)_t \\
&= \cos(x)\cos(y) - \sin(x)\sin(y) \\
&= \cos(x+y)
\end{aligned}$$

$$\begin{aligned}
f_{ts} &= (f_t)_s = \frac{\partial f_t}{\partial s} \\
&= \left(\frac{\partial f_t}{\partial x}\right)\left(\frac{\partial x}{\partial s}\right) + \left(\frac{\partial f_t}{\partial y}\right)\left(\frac{\partial y}{\partial s}\right) \\
&= \left(\cos(x+y)\right)_x (t+s)_s + \left(\cos(x+y)\right)_t (t-s)_s \\
&= -\sin(x+y)(1) - \sin(x+y)(-1) \\
&= 0
\end{aligned}$$