LECTURE 14: SERIES (II)

Let's continue our series extravaganza! Today's goal is to prove the celebrated Ratio, Root, and Integral Tests

1. The Root Test

Video: Root Test Proof

Example 1:

Use the root test to figure out if the following series converges:

$$\sum_{n=0}^{\infty} \frac{n}{3^n}$$

Let $a_n = \frac{n}{3^n}$, then the root test tells you to look at:

$$|a_n|^{\frac{1}{n}} = \left|\frac{n}{3^n}\right|^{\frac{1}{n}} = \frac{n^{\frac{1}{n}}}{3^{n\left(\frac{1}{n}\right)}} = \frac{n^{\frac{1}{n}}}{3} \overset{n\to\infty}{\to} \frac{1}{3} = \alpha < 1$$

Therefore $\sum a_n$ converges absolutely.

Since $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ doesn't always exist, we need to replace this with $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ (which always exists). We then obtain the root test:

Date: Thursday, October 14, 2021.

Root Test

Consider $\sum a_n$ and let

$$\alpha = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

(1) If $\alpha < 1$, then $\sum a_n$ converges absolutely (that is $\sum |a_n|$ converges)

(2) If $\alpha > 1$, then $\sum a_n$ diverges

(3) If $\alpha = 1$, then the root test is inconclusive, meaning that you'd have to use another test

Proof of (1): ($\alpha < 1 \Rightarrow$ converges absolutely)

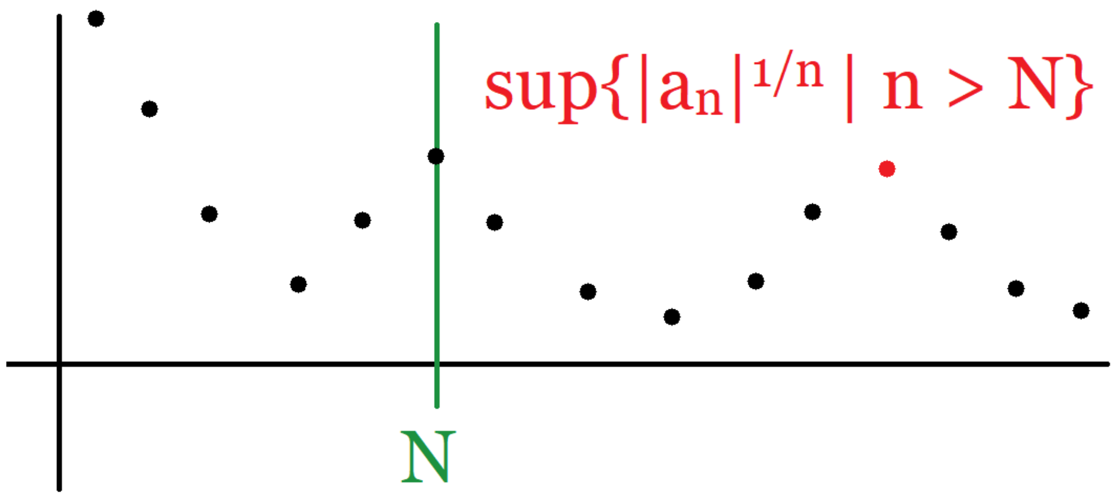
Main Idea: Since $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \alpha < 1$, then for large n we have $|a_n|^{\frac{1}{n}} \leq \alpha$. So $|a_n| \leq \alpha^n$ and therefore $\sum |a_n| \leq \sum \alpha^n$, which is a geometric series that converges, since $\alpha < 1$.

We now need to make this precise:

Since $\alpha < 1$, let $\epsilon > 0$ be such that $\alpha < \alpha + \epsilon < 1$ (need some wiggle room between α and 1)

By definition of lim sup, we have

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{N\to\infty} \sup \left\{ |a_n|^{\frac{1}{n}} \mid n > N \right\} = \alpha$$



Hence, by definition of a limit, there is N_1 such that if $N > N_1$, then

$$\begin{aligned}\left|\sup\left\{|a_n|^{\frac{1}{n}} \mid n > N\right\} - \alpha\right| &< \epsilon \\ \Rightarrow \sup\left\{|a_n|^{\frac{1}{n}} \mid n > N\right\} - \alpha &< \epsilon \\ \Rightarrow \sup\left\{|a_n|^{\frac{1}{n}} \mid n > N\right\} &< \alpha + \epsilon\end{aligned}$$

But then, by definition of sup (think max), for all $n > N$, we have:

$$|a_n|^{\frac{1}{n}} < \alpha + \epsilon \Rightarrow |a_n| < (\alpha + \epsilon)^n$$

And, in particular:

$$\sum_{n=N+1}^{\infty} |a_n| \leq \sum_{n=N+1}^{\infty} (\alpha + \epsilon)^n = \sum_{n=1}^{\infty} r^n$$

Where $r = \alpha + \epsilon < 1$. But the latter is just a geometric series with $|r| < 1$ and therefore converges. Hence, by the comparison test,

$$\sum_{n=N+1}^{\infty} |a_n| \text{ converges}$$

And so, ignoring the first couple of terms, $\sum a_n$ converges absolutely ✓

Proof of (2): ($\alpha > 1 \Rightarrow$ diverges)

Even easier! Remember that for any sequence (s_n), there is a subsequence (s_{n_k}) converging to $\limsup_{n\to\infty} s_n$.

Therefore here there is a subsequence $|a_{n_k}|^{\frac{1}{n_k}}$ of $|a_n|^{\frac{1}{n}}$ converging to $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \alpha > 1$

But this means that for all k large enough, we must have

$$|a_{n_k}|^{\frac{1}{n_k}} > 1 \Rightarrow |a_{n_k}| > 1^{n_k} = 1$$

But since $|a_{n_k}| > 1$ for every k, we *cannot* have $a_n \to 0$. Therefore $a_n \nrightarrow 0$, and so $\sum a_n$ diverges by the divergence test. ✓

Proof of (3): All we need to do is find two series with $\alpha = 1$, one which converges absolutely, and the other one which diverges.

Consider $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges since it's a $1-$series, and

$$|a_n|^{\frac{1}{n}} = \left(\frac{1}{n}\right)^{\frac{1}{n}} = \frac{1}{n^{\frac{1}{n}}} \to \frac{1}{1} = 1$$

So $\alpha = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 1$.

Now consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges absolutely since it's a $2-$series, and

$$|a_n|^{\frac{1}{n}} = \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = \frac{1}{n^{\frac{2}{n}}} = \frac{1}{\left(n^{\frac{1}{n}}\right)^2} \to \frac{1}{1} = 1$$

So $\alpha = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 1$ □

2. The Ratio Test

Video: Ratio Test Proof

On the other side of the spectrum is the ratio test:

Example 2:

Use the ratio test to figure out if the following series converges:

$$\sum_{n=0}^{\infty} \frac{n}{3^n}$$

This time look at ratios of successive terms:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \left(\frac{3^n}{3^{n+1}}\right)\left(\frac{n+1}{n}\right) = \left(\frac{1}{3}\right)\left(\frac{n+1}{n}\right) \to \frac{1}{3} < 1$$

Therefore the series converges absolutely.

Note: The ratio test is **excellent** for series involving $n!$, like $\sum \frac{1}{n!}$

Here again, since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ might not exist, we need to replace the limit with $\limsup$ and $\liminf$:

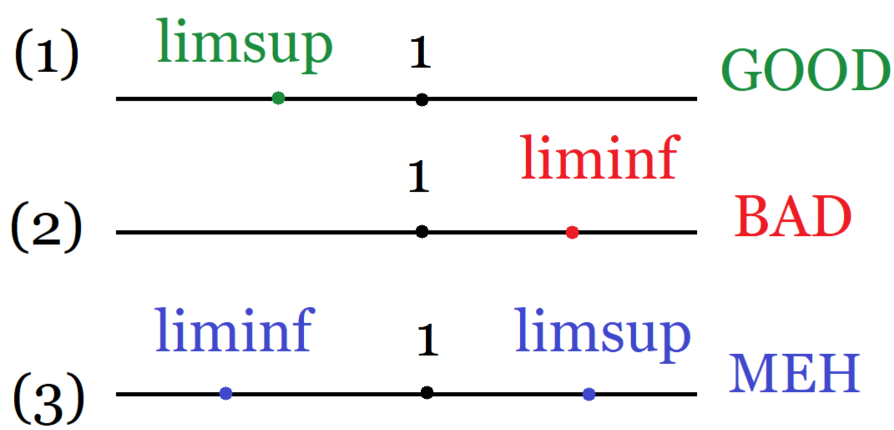
Ratio Test:

Consider $\sum a_n$. Then:

(1) If $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ converges absolutely.

(2) If $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum a_n$ diverges

(3) If $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, then the ratio test is inconclusive.



Proof: Muuuuuch easier than the proof of the root test, since we've already done the hard part in section 12 ☺

Recall: Pre-Ratio Test

$$\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \leq \liminf_{n\to\infty}|a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty}|a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

(1) If $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then, in the above, we get

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

$$\text{So} \quad \alpha =: \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$$

And therefore by the **root** test, $\sum a_n$ converges absolutely ✓

(2) If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then, in the above, we get:

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \geq \liminf_{n\to\infty} |a_n|^{\frac{1}{n}} \geq \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$$

$$\text{So} \quad \alpha =: \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} > 1$$

And hence by the **root** test, $\sum a_n$ diverges. ✓

(3) Finally, just as before, we need to find two series $\sum a_n$ with $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, one of which being divergent and the other one absolutely convergent.

On the one hand $\sum \frac{1}{n}$, which is divergent, since it's a $1-$series, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \to 1$$

$$\text{Therefore} \quad \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \leq 1 \leq 1 = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Now consider $\sum \frac{1}{n^2}$, which is absolutely convergent, since it's a 2−series, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \left(\frac{n}{n+1}\right)^2 \to 1$$

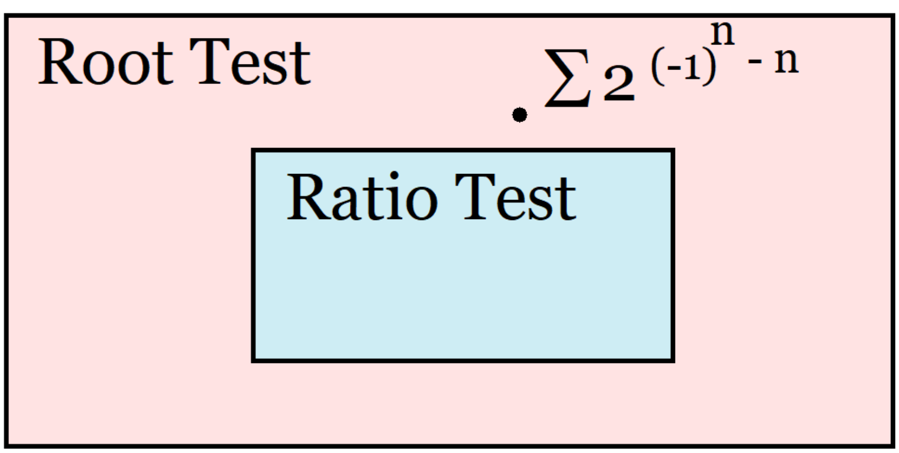
Therefore $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \leq 1 \leq 1 = \liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ □

Summary:

The Root test is strictly better than the ratio test:

If $\sum a_n$ converges (or diverges) by the ratio test, then it converges (or diverges) by the root test as well.

But there are examples of series (like the one below) which converge (or diverge) by the root test, but for which the ratio test is inconclusive.



3. Root Test > Ratio Test

Video: Ratio Test Vs Root Test

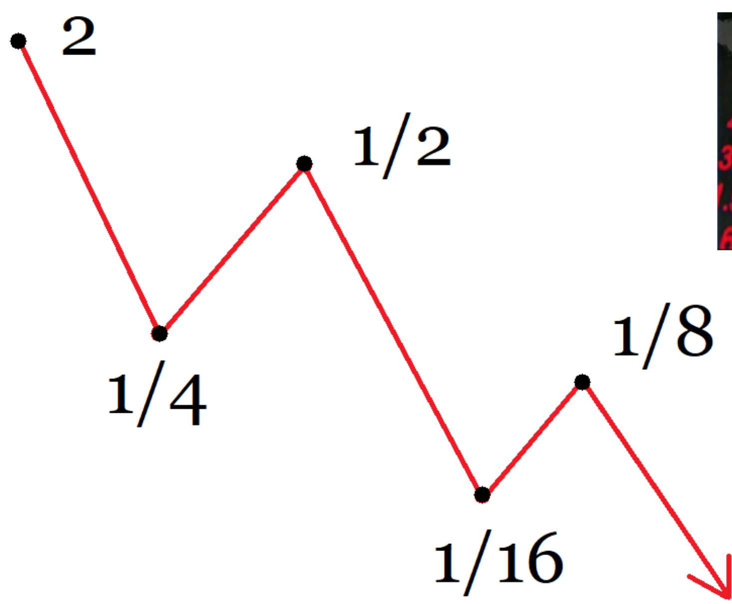
As another illustration of why the root test is better than the ratio test, consider the following:

Example 3:

Does the following series converge?

$$\sum_{n=0}^{\infty} 2^{(-1)^n - n} = 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \dots$$

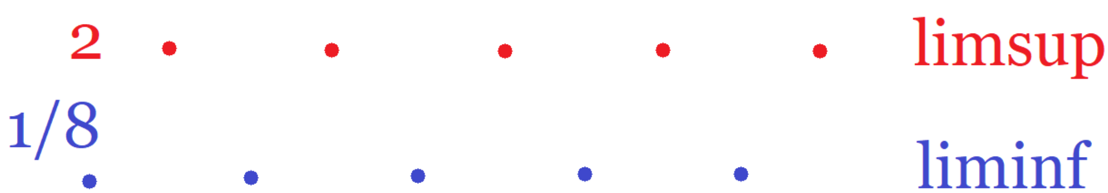
This is what I like to call the stock market series, or the *Not Stonks* series:



Let's try to apply both the ratio test and the root test to this series, in order to see who wins.

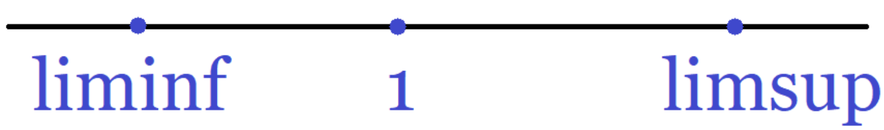
Ratio Test:

$$
\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{2^{(-1)^{n+1}-(n+1)}}{2^{(-1)^n-n}} \\
&= 2^{(-1)^{n+1}-\cancel{n}-1-(-1)^n+\cancel{n}} \\
&= 2^{-(-1)^n-(-1)^n-1} \\
&= 2^{-((-1)^n+(-1)^n+1)} \\
&= 2^{-2(-1)^n-1} \\
&= \left(\frac{1}{8}, 2, \frac{1}{8}, 2, \frac{1}{8}, 2, \dots\right)
\end{aligned}
$$



Therefore $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{8}$ and $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2$ and so:

$$
\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|
$$



So we are in the third case of the ratio test, and so the ratio test is **inconclusive**.

Root Test:

$$|a_n|^{\frac{1}{n}} = \left(2^{(-1)^n - n}\right)^{\frac{1}{n}} = 2^{\left(\frac{(-1)^n}{n}\right) - 1} \to 2^{0-1} = 2^{-1} = \frac{1}{2} < 1$$

($\frac{(-1)^n}{n} \to 0$ follows from the squeeze theorem, since it is squeezed between $-\frac{1}{n}$ and $\frac{1}{n}$)

$$\text{Hence} \quad \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{2} < 1$$

And therefore by the root test, $\sum a_n$ converges absolutely.

4. Root Test Pitfall

Video: Root Test Pitfall

That said, don't get *too* overexcited, the root test doesn't *always* work. In particular, don't think that just because you see something to the power of n, you have to apply the root test!

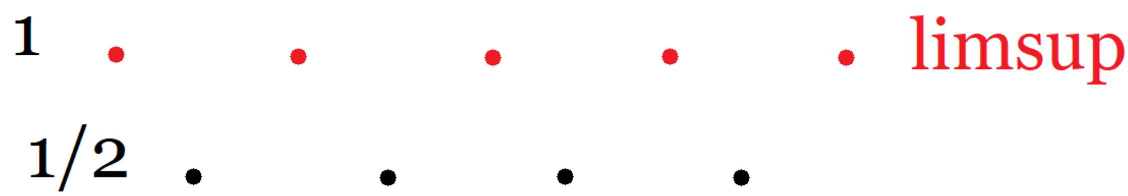
Example 4:

Does the following series converge?

$$\sum_{n=0}^{\infty} \left(\frac{2}{(-1)^n - 3}\right)^n$$

First try: Let's try using the root test:

$$|a_n|^{\frac{1}{n}} = \left|\frac{2}{(-1)^n - 3}\right| = \left(1, \frac{1}{2}, 1, \frac{1}{2}, \dots\right)$$



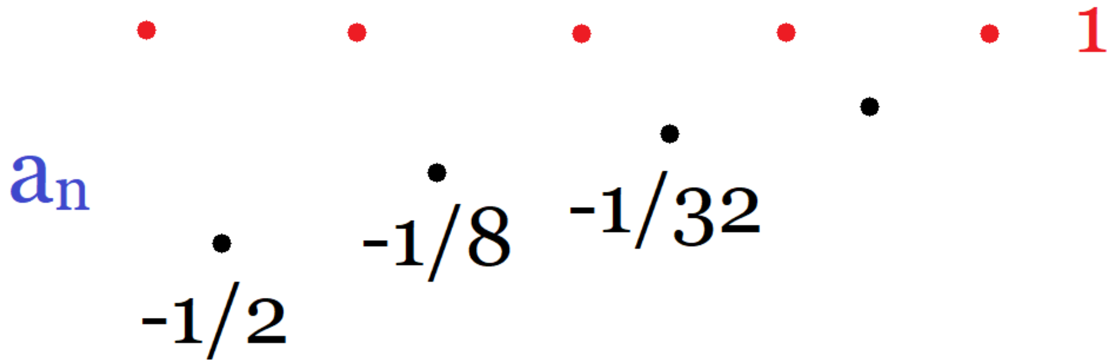
$$\alpha = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 1$$

So the root test is inconclusive, and we'll have to try another method.

Note: The ratio test would also be inconclusive (by the pre-ratio test), so we'll have to try to find another way of doing this:

Second try: Look at the sequence (a_n) itself!

$$a_n = \left(\frac{2}{(-1)^n - 3}\right)^n = \left(1, -\frac{1}{2}, 1, -\frac{1}{8}, 1, -\frac{1}{32}, 1, -\frac{1}{128}, 1, \dots\right)$$



Notice that every other term of a_n is 1, hence $a_n \nrightarrow 0$, and therefore $\sum a_n$ diverges by the divergence test.

5. Integral Test 1

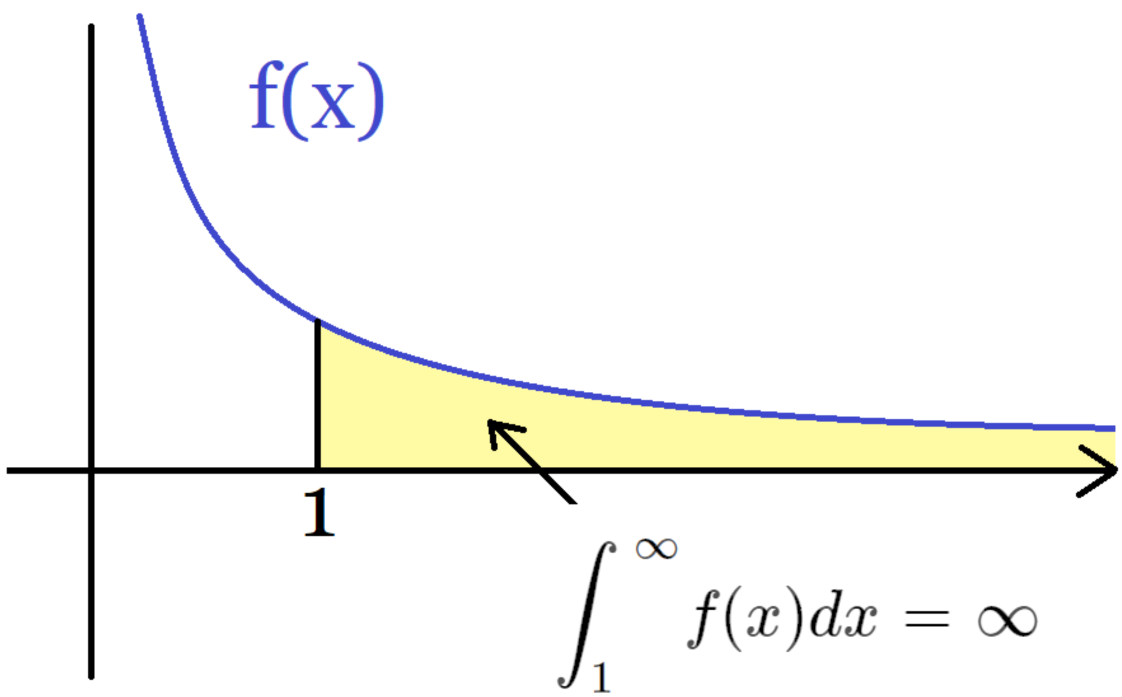
Video: Integral Test 1

This final test is *integral* in our understanding of series! It basically says that if an integral is ∞, then the corresponding series is ∞ as well.

Integral Test 1:

Suppose $f(x) \geq 0$ is decreasing on $[1, \infty)$, then

$$\int_1^\infty f(x)dx = \infty \Rightarrow \sum_{n=1}^{\infty} f(n) \text{ diverges}$$



Example 5:

Does the 1−series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Let $f(x) = \frac{1}{x}$ (so $f(n) = \frac{1}{n}$), then

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x} dx = [\ln(x)]_1^{\infty} = \ln(\infty) - \ln(1) = \infty - 0 = \infty$$

(We're being a bit hand-wavy here because we haven't defined improper integrals, but the result is still the same)

Therefore, by the integral test, $\sum \frac{1}{n}$ diverges.

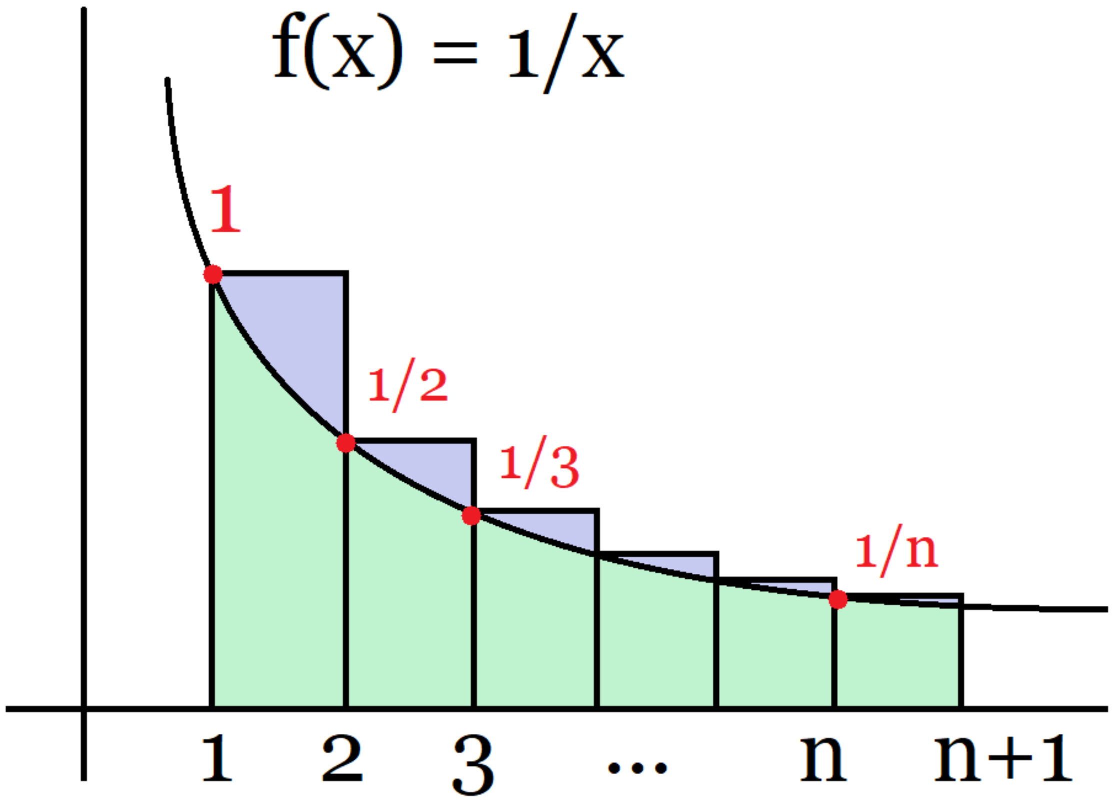
Proof:

Note: To make things a bit easier to understand, we will do the proof for $f(x) = \frac{1}{x}$ and show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. The *exact* same proof works if you simply replace $\frac{1}{x}$ by $f(x)$ (see Homework)

Consider the partial sums:

$$s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Main Idea: Interpret the sum above in terms of areas of rectangles, and compare it with the area under f, that is $\int_1^{\infty} \left(\frac{1}{x}\right) dx$.



Start with the rectangle with base $[1, 2]$ and height $f(1) = 1$ (left endpoint), which has area $1 \times 1 = 1$.

Then consider the rectangle with base $[2, 3]$ and height $f(2) = \frac{1}{2}$, which has area $1 \times \frac{1}{2} = \frac{1}{2}$

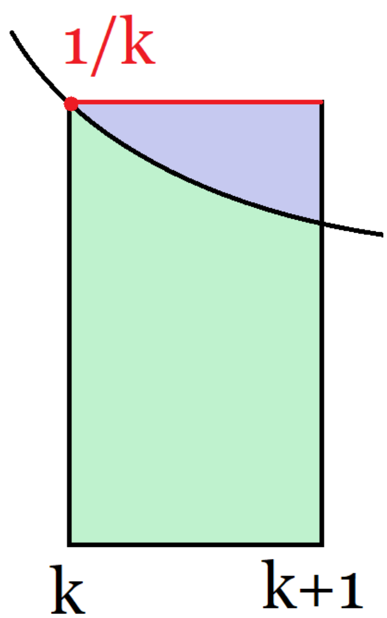
Continue that way until you have the rectangle with base $[n, n+1]$ and height $\frac{1}{n}$, which has area $\frac{1}{n}$

$$\text{Then} \quad s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \text{ Sum of areas of } n \text{ rectangles}$$

(In the picture, s_n is the sum of the green and the blue regions)

On the other hand, the sum of the areas is *larger* than the area under f from 1 to $n+1$ which is $\int_1^{n+1} f(x)dx$. (see the picture above).

This is because f is decreasing, and therefore on each interval $[k, k+1]$ (with $k = 1, \ldots, n$), the left-endpoint is larger than any other value of f, and therefore the area of each rectangle is larger than the area under f on $[k, k+1]$.



$$\text{And therefore } \quad s_n = \sum_{k=1}^{n} \frac{1}{k} \geq \int_1^{n+1} f(x)dx$$

$$\text{But } \lim_{n\to\infty} \int_1^{n+1} f(x)dx = \int_1^{\infty} f(x)dx = \infty \quad \text{(By assumption)}$$

Therefore, by comparison, $\lim_{n\to\infty} s_n = \infty$, that is $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ □

Corollary:

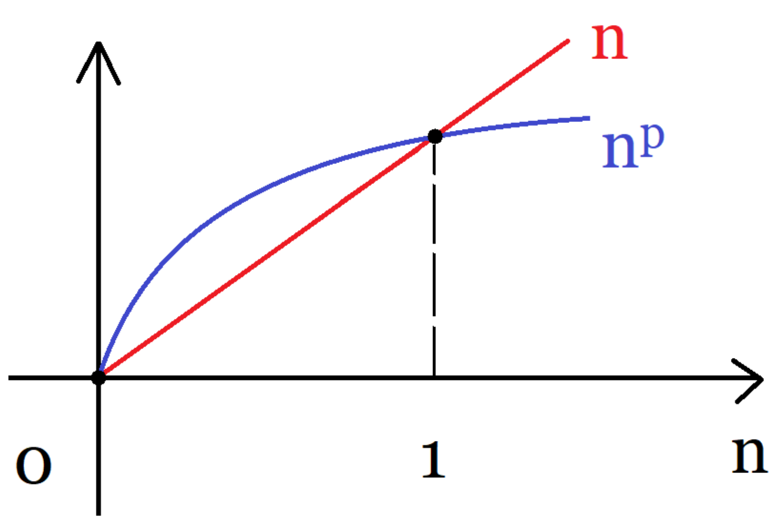
If $p < 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{diverges}$$

Example 6:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} = \infty$$

Proof: Either use the integral test, or notice that if $p < 1$, then, since $n \geq 1$, we have $n^p \leq n$ (Think $\sqrt{n} \leq n$)



$$\text{Therefore} \quad \sum_{n=1}^{\infty} \frac{1}{n^p} \geq \sum_{n=1}^{\infty} \frac{1}{n}$$

But since $\sum \frac{1}{n} = \infty$, we get $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$ by the comparison test.