LECTURE 15: SERIES (III) + CONTINUITY (I)

1. Integral Test 2

Video: Integral Test 2

Welcome to the dark side of the Integral Test, where we use it to show that a series converges:

Example 1:

Does the 2−series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Let $f(x) = \frac{1}{x^2}$, then

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^{\infty} = -\frac{1}{\infty} + \frac{1}{1} = 1 < \infty$$

Therefore $\sum \frac{1}{n^2}$ converges.

Note: In fact, one can show $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Check out this video for a proof if you're interested: Sum of $\frac{1}{n^2}$

Date: Tuesday, October 19, 2021.

WARNING: In general, the value of the integral tells us nothing about the value of the series. For instance, here $\int_1^\infty \frac{1}{x^2}dx = 1$, but $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$, which are not related.

Integral Test 2:

Suppose $f(x) \geq 0$ and is decreasing on $[1, \infty)$. Then

$$\int_1^\infty f(x)dx \text{ converges } \Rightarrow \sum_{n=1}^\infty f(n) \text{ converges}$$

Proof: This time we'll illustrate the proof with $\frac{1}{x^2}$.

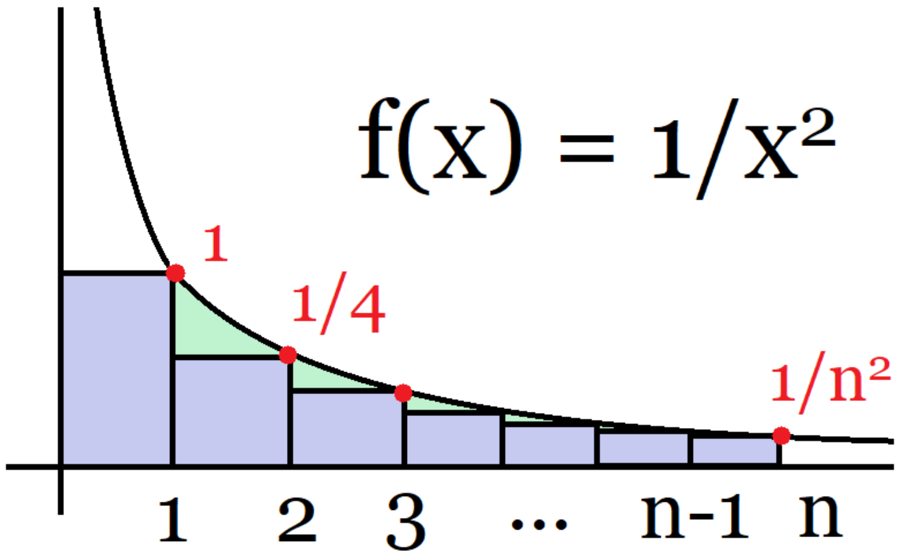
Consider again the partial sums

$$s_n = \sum_{k=1}^n \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$

It is enough to show that (s_n) is bounded.

This time, instead of taking the left endpoints, we take right endpoints:

- Start with the rectangle with base $[0, 1]$ and height $f(1) = 1$
- Then consider the rectangle with base $[1, 2]$ and height $f(2) = \frac{1}{4}$
- Continue this way until you get the rectangle with base $[n-1, n]$ and height $f(n) = \frac{1}{n^2}$

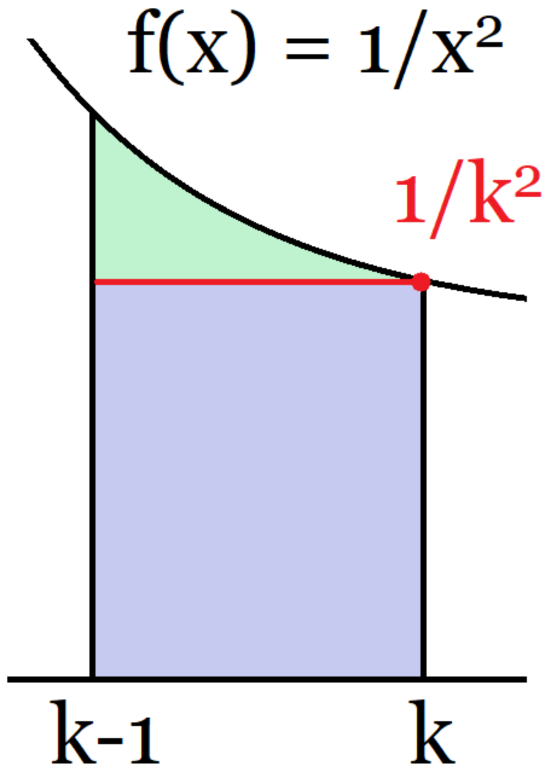


$$s_n = 1 + \frac{1}{4} + \cdots + \frac{1}{n^2} = \text{ Sum of the areas of the rectangles}$$

Note: Since $\int_1^\infty f(x)dx$ is only defined on $[1, \infty)$, we need to ignore the first rectangle:

$$s_n = 1 + \text{ (Rectangles 2 to n)}$$

This time, notice that the area under the graph of f from 1 to n is **bigger** than the sum of the areas of rectangles 2 to n.



This is again because f is decreasing: On each rectangle $[k-1,k]$, $f(k)=\frac{1}{k^2}$ is the smallest value of f on the rectangle. Therefore:

$$\begin{aligned} s_n &\leq 1 + \text{ Area of Rectangles 2 to n} \\ &\leq 1 + \int_1^n f(x)dx \\ &\leq 1 + \int_1^\infty f(x)dx \text{ (since } f \geq 0) \end{aligned}$$

Therefore, with $M =: 1 + \int_1^\infty f(x)dx$ we get $0 \leq s_n \leq M$

Hence $|s_n| \leq M$ for all n, and so (s_n) is bounded, and therefore $\sum \frac{1}{n^2}$ converges $\square$

Corollary:

If $p > 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges}$$

Proof: This is because

$$\int_1^\infty \frac{1}{x^p} dx = \int_1^\infty x^{-p} dx = \left[\frac{x^{1-p}}{1-p}\right]_1^\infty = 0 - \left(\frac{1}{1-p}\right) = \frac{1}{p-1} < \infty$$

Therefore, $\sum \frac{1}{n^p}$ converges by the Integral Test.

Combining this with the corollary from last time, we get:

Corollary: [p-series]

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges } \Leftrightarrow p > 1$$

2. Alternating Series Test

Video: Alternating Series Test

Finally, there is a wonderful test called the alternating series test, which *basically* says that all alternating series converge.

Definition:

An **alternating series** is a series of the form $\sum(-1)^n a_n$ or $\sum(-1)^{n+1} a_n$, where $a_n \geq 0$

Example 2:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is an *alternating* series with $a_n = \frac{1}{n}$

In other words, an alternating series alternates between positive and negative values

Alternating Series Test:

If $a_n \geq 0$, is decreasing and converges to 0 then $\sum(-1)^n a_n$ converges

Example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges because $a_n = \frac{1}{n} \geq 0$ is decreasing and converges to 0.

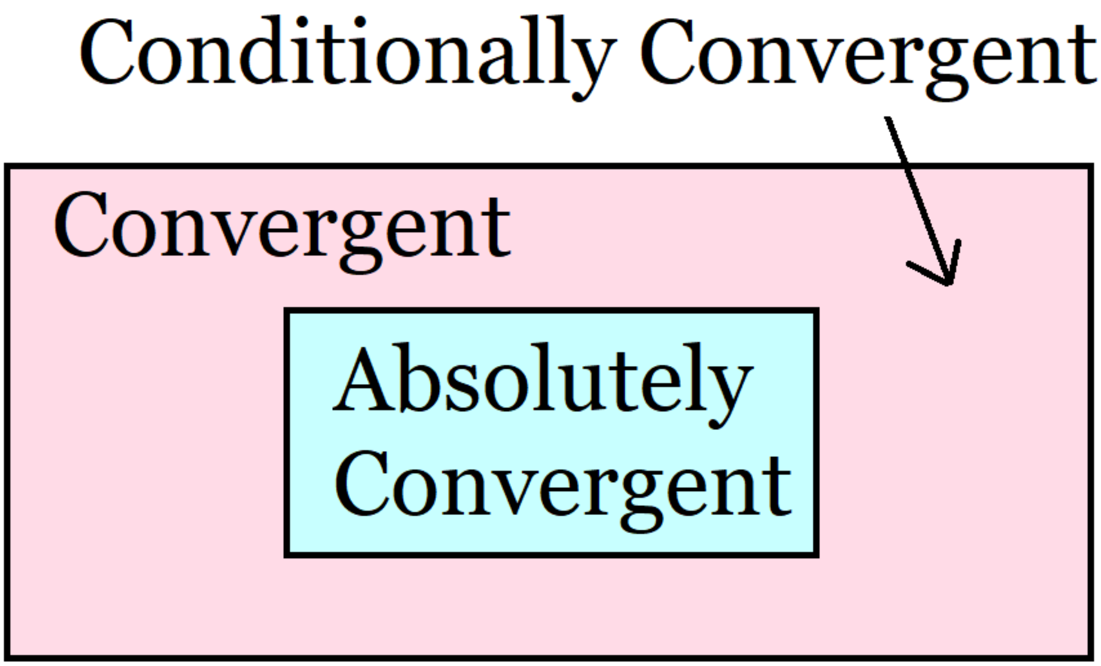
Proof: (Optional, see Appendix at the end)

The alternating series test is useful to show that a series converges conditionally:

Recall:

(1) $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent

(2) Absolute Convergence $\Rightarrow$ Convergence



Series that are convergent, but not absolutely convergent are called *conditionally convergent*

Definition:

$\sum a_n$ is **conditionally** convergent if $\sum a_n$ converges, but $\sum |a_n|$ diverges

Example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \dots$$

This series is convergent by the alternating series test (see above), but

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

is divergent, and therefore $\sum \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Optional Fun Fact:

If $\sum a_n$ is conditionally convergent, then you can rearrange $\sum a_n$ to get any limit you want!

For example, a rearrangement of

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots\dots$$

$$\text{is } 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$$

Not only can it be shown that the two series here have two different limits, but given any limit, you can rearrange the first series to make it converge to that limit (WOW), see this video if you're interested: Riemann Series Theorem

This concludes our series extravaganza, and now welcome to the third chapter of our Analysis Adventure! In this chapter, we'll explore the magical world of continuous functions.

3. Example 1: The Basics

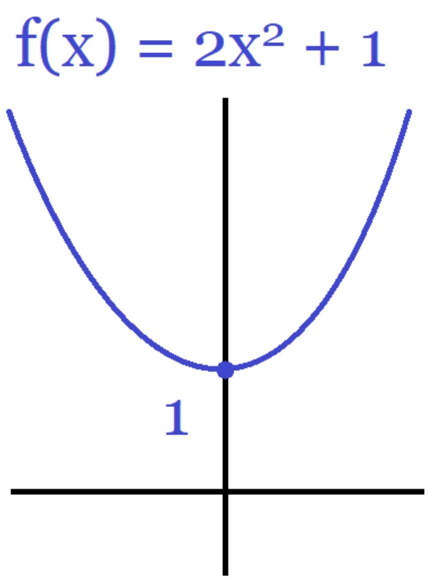
Video: Example 1: The Basics

Example 1:

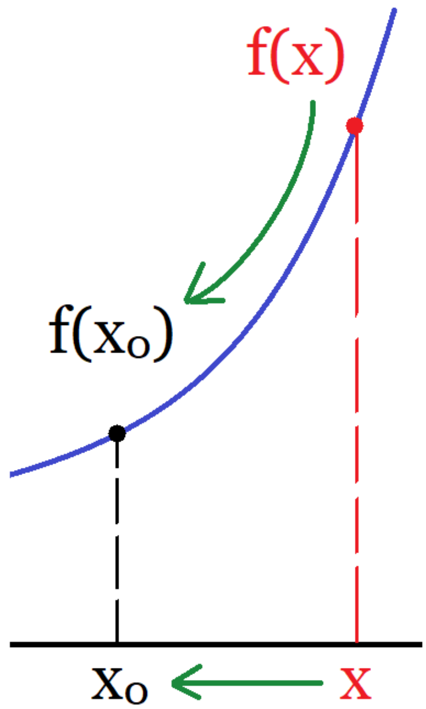
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 2x^2 + 1$$

Show that f is continuous (at x_0)



Intuitively: f continuous at x_0 means that if x goes to x_0, then $f(x)$ goes to $f(x_0)$. Or, in other words, if the inputs x and x_0 are close, then the outputs $f(x)$ and $f(x_0)$ are close as well.



What does *goes to* mean? It turns out that there are two different ways of interpreting it, which are equally useful (and in fact, we'll use both of them in this chapter).

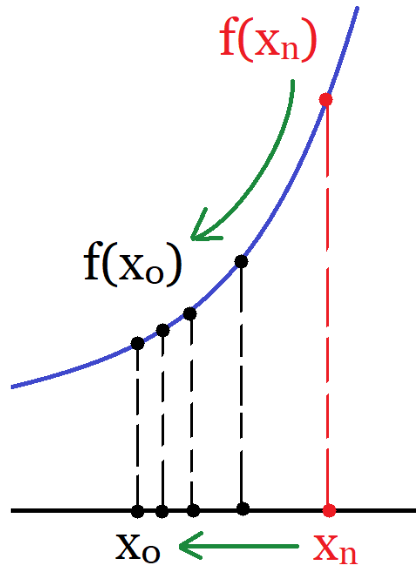
Definition 1:

f is **continuous at** x_0 if: whenever x_n is a sequence that converges to x_0, then $f(x_n)$ converges to $f(x_0)$

f is **continuous** if: for all x_0 (in the domain of f), f is continuous at x_0

$$\text{In other words: } x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$$

That is, if the inputs x_n get closer to x_0, then so do the outputs $f(x_n)$ to $f(x_0)$.



Using this definition, show that $f(x) = 2x^2 + 1$ is continuous at x_0

Proof: Easy, because we have done the hard work in section 9!

Suppose $x_n \to x_0$, then, from section 9, we know that $(x_n)^2 \to (x_0)^2$, and therefore $2(x_n)^2 \to 2(x_0)^2$, and hence $2(x_n)^2 + 1 \to 2(x_0)^2 + 1$, that is $f(x_n) \to f(x_0)$ ✓

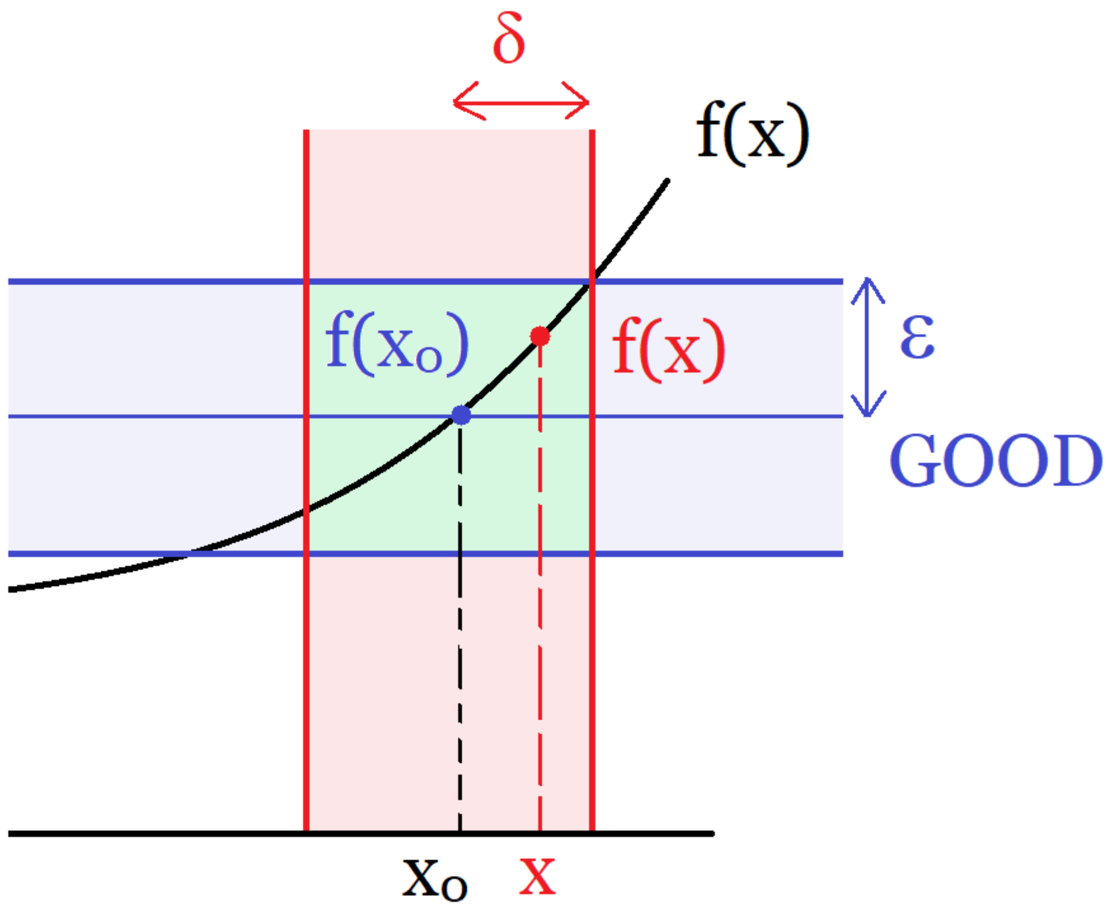
Even though this definition is nice, there are some cases where it is better to use a more *direct* definition of continuity that doesn't use sequences. This is called the (celebrated) $\epsilon - \delta$ definition of continuity:

Definition 2:

f is **continuous** at x_0 if for all $\epsilon > 0$ there is $\delta > 0$ such that for all x, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$

f is **continuous** if for all x_0, f is continuous at x_0

This should be read as: No matter how small the error ϵ, there is some threshold δ such that, if $|x - x_0|$ is within the threshold δ, then $|f(x) - f(x_0)|$ is within the good region ϵ.



Interpretation: In the picture above, no matter how small the good region (blue), if x is close enough to x_0 (red), then $f(x)$ is guaranteed to be in the good region (blue/green)

Note: Compare this to the definition of the limit of a sequence (for all $\epsilon > 0$ there is N such that if $n > N$ then $|s_n - s| < \epsilon$). N is large because we want $n \to \infty$, but here δ is small because we want x to be close to x_0.

Using that new definition, show that $f(x) = 2x^2 + 1$ is continuous at x_0

Proof: Just like limits of sequences, we need to do it in two steps, with scratchwork and the actual proof.

STEP 1: Scratchwork: Calculate

$$\begin{aligned} |f(x) - f(x_0)| &= \left|2x^2 + 1 - \left(2(x_0)^2 + 1\right)\right| \\ &= \left|2x^2 + \cancel{1} - 2(x_0)^2 - \cancel{1}\right| \\ &= 2\left|x^2 - (x_0)^2\right| \\ &= 2|x - x_0||x + x_0| \\ &\overset{?}{<} \epsilon \end{aligned}$$

Note: The $|x - x_0|$ term is *good* because we can control it, make it less than δ. It's the $|x + x_0|$ that we need to make independent of x (because δ shouldn't depend on x). For this, remember x is close to x_0:

Suppose $|x - x_0| < 1$ (remember this for **STEP 2**), then we have:

$$|x + x_0| \le |x| + |x_0| = |x - x_0 + x_0| + |x_0| \le |x - x_0| + |x_0| + |x_0| < 1 + 2|x_0|$$

This is muuuuch better because this is constant, independent of x

In conclusion, putting everything together, we get:

$$|f(x) - f(x_0)| = 2|x - x_0||x + x_0| \le 2|x - x_0|(2|x_0| + 1) < \epsilon$$

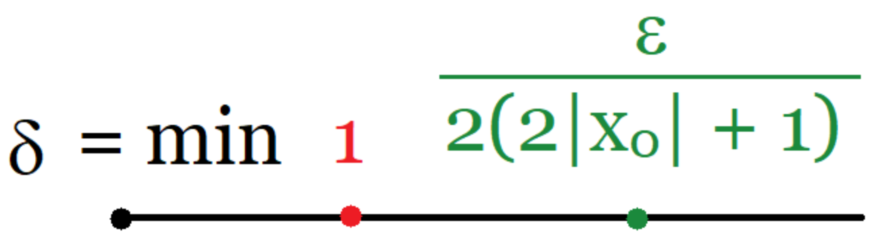
$$\text{Which gives } |x - x_0| < \frac{\epsilon}{2(2|x_0| + 1)}$$

Which suggests to let $\delta = \frac{\epsilon}{2(2|x_0|+1)}$, except we need to remember that we also had $|x - x_0| < 1$

STEP 2: Let $\epsilon > 0$ be given, and let

$$\delta = \min\left\{1, \frac{\epsilon}{2\left(2\left|x_0\right| + 1\right)}\right\}$$

(Before we had $N = \max$ but now we need to let $\delta = \min$ because we want δ to be small)



If $|x - x_0| < \delta$, then, on the one hand, we have $|x - x_0| < 1$, and so $|x + x_0| < 2\left|x_0\right| + 1$. On the other hand, $|x - x_0| < \frac{\epsilon}{2(2|x_0|+1)}$, and therefore

$$\begin{aligned} |f(x) - f(x_0)| &= 2\left|x - x_0\right|\left|x + x_0\right| \\ &\leq 2\left|x - x_0\right|\left(2\left|x_0\right| + 1\right) \\ &< 2\left(\frac{\epsilon}{2\cancel{\left(2\left|x_0\right| + 1\right)}}\right)\cancel{\left(2\left|x_0\right| + 1\right)} \\ &= \epsilon \checkmark \quad \square \end{aligned}$$

Note: Of course the sequence approach is much faster here, but you really need to be comfortable with proving continuity both ways.

4. Example 2: $x^2 \sin\left(\frac{1}{x}\right)$ and $\frac{1}{x^2}\sin\left(\frac{1}{x}\right)$

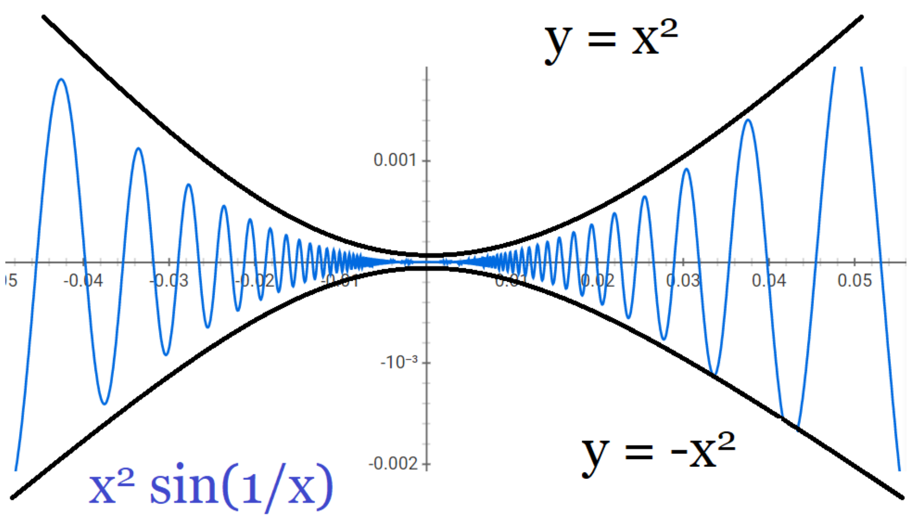
Video: Example 2: $x^2 \sin\left(\frac{1}{x}\right)$ and $\frac{1}{x^2}\sin\left(\frac{1}{x}\right)$

In fact, here are two examples; one illustrating that the $\epsilon - \delta$ approach is better, and the other one showing that the sequence approach is better.

Example 2a:

$$\text{Let } f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Show that f is continuous at 0



(The plot is courtesy Google Plots)

Notice that f is squeezed between the parabolas $y = -x^2$ and $y = x^2$, so intuitively it looks like, as x goes to 0, $f(x)$ goes to $f(0) = 0$, but the whole point is to make this rigorous! For this, we'll use the $\epsilon - \delta$ definition of continuity.

STEP 1: Scratch Work

$$|f(x) - f(0)| = \left|x^2 \sin\left(\frac{1}{x}\right) - 0\right| = \left|x^2 \sin\left(\frac{1}{x}\right)\right| = x^2 \underbrace{\left|\sin\left(\frac{1}{x}\right)\right|}_{\leq 1} = x^2 < \epsilon$$

Which gives $-\sqrt{\epsilon} < x < \sqrt{\epsilon}$, so $|x| < \sqrt{\epsilon}$. This suggests to let $\delta = \sqrt{\epsilon}$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, then if $\delta = \sqrt{\epsilon}$, if $|x - 0| = |x| < \delta = \sqrt{\epsilon}$, then

$$|f(x) - f(0)| = x^2 \left|\sin\left(\frac{1}{x}\right)\right| \leq x^2 < \left(\sqrt{\epsilon}\right)^2 = \epsilon \checkmark$$

Hence f is continuous at 0.

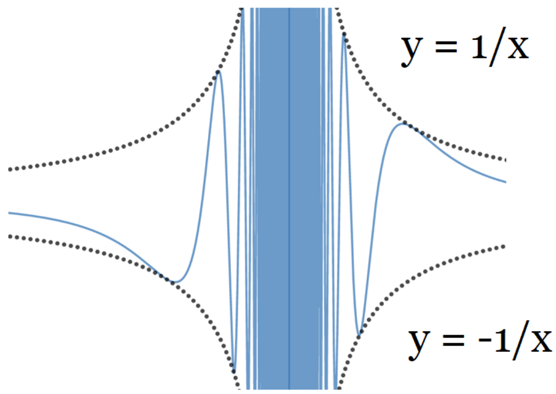
Note: It's also possible to use the squeeze theorem (from calculus) for this, see section 19

On the other hand, let's now do an example where the sequence definition works better:

Example 2b:

$$\text{Let } f(x) = \begin{cases} \frac{1}{x}\sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Show that f is **not** continuous at 0



(Graph courtesy Desmos)

Here, to show f is not continuous at 0, we need to find a sequence $x_n \to 0$ but such that

$$f(x_n) = \frac{1}{x_n} \sin\left(\frac{1}{(x_n)^2}\right) \nrightarrow 0$$

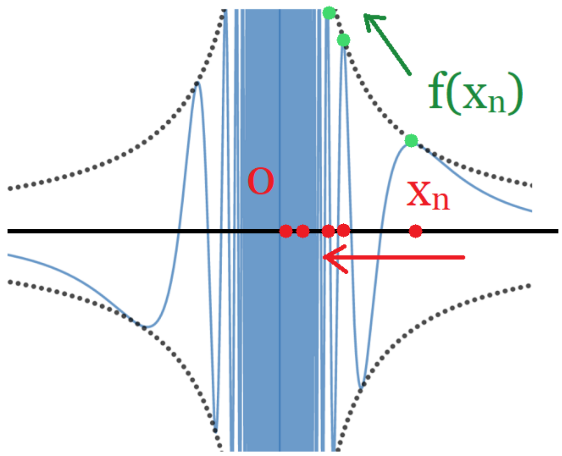
To simplify the task, let's choose $x_n \to 0$ such that

$$\sin\left(\frac{1}{(x_n)^2}\right) = 1 \Rightarrow \frac{1}{(x_n)^2} = \frac{\pi}{2} + 2\pi n \Rightarrow (x_n)^2 = \frac{1}{\frac{\pi}{2} + 2\pi n} \Rightarrow x_n = \frac{\pm 1}{\sqrt{\frac{\pi}{2} + 2\pi n}}$$

So for instance, if (x_n) is the sequence $x_n = \frac{1}{\sqrt{\frac{\pi}{2}+2\pi n}}$, then

$$x_n = \frac{1}{\sqrt{\frac{\pi}{2} + 2\pi n}} \to \frac{1}{\sqrt{\frac{\pi}{2} + \infty}} = \frac{1}{\infty} = 0$$

$$\text{So } x_n \to 0 \text{ but } f(x_n) = \frac{1}{x_n} \underbrace{\sin\left(\frac{1}{(x_n)^2}\right)}_{=1} = \frac{1}{x_n} = \sqrt{\frac{\pi}{2} + 2\pi n} \to \infty$$



So $f(x_n)$ doesn't go to $f(0) = 0$. Hence f is not continuous at 0.

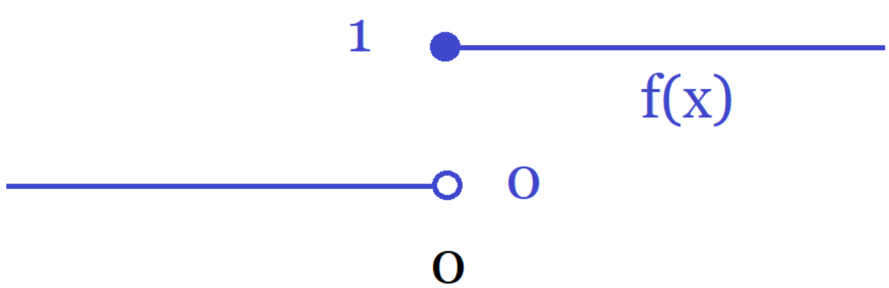
5. Example 3: Not Continuous

Video: Example 3: Not Continuous

Example 3:

$$\text{Let } f(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$$

Show that f is **not** continuous at 0



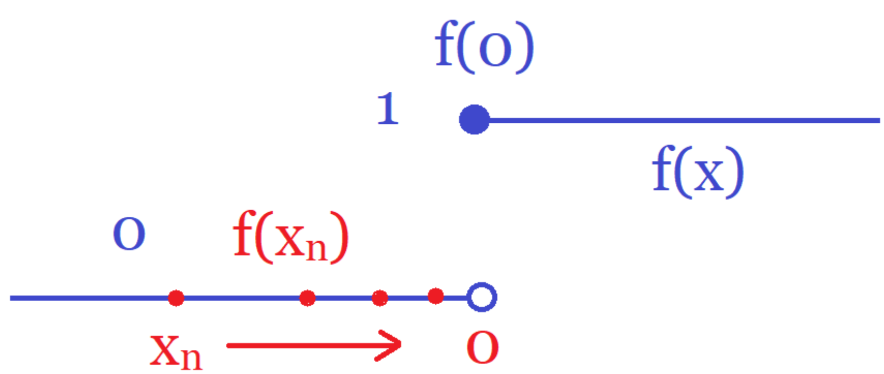
Here we'll do it both ways, to really illustrate the differences between the two approaches.

The Sequence Way:

Let (x_n) be any sequence of negative numbers converging to 0. For example, let $x_n = -\frac{1}{n}$, then $x_n \to 0$, but since $x_n < 0$, we have $f(x_n) = 0$, and so

$$f(x_n) = 0 \to 0 \neq f(0) = 1$$

Therefore f is not continuous at 0.



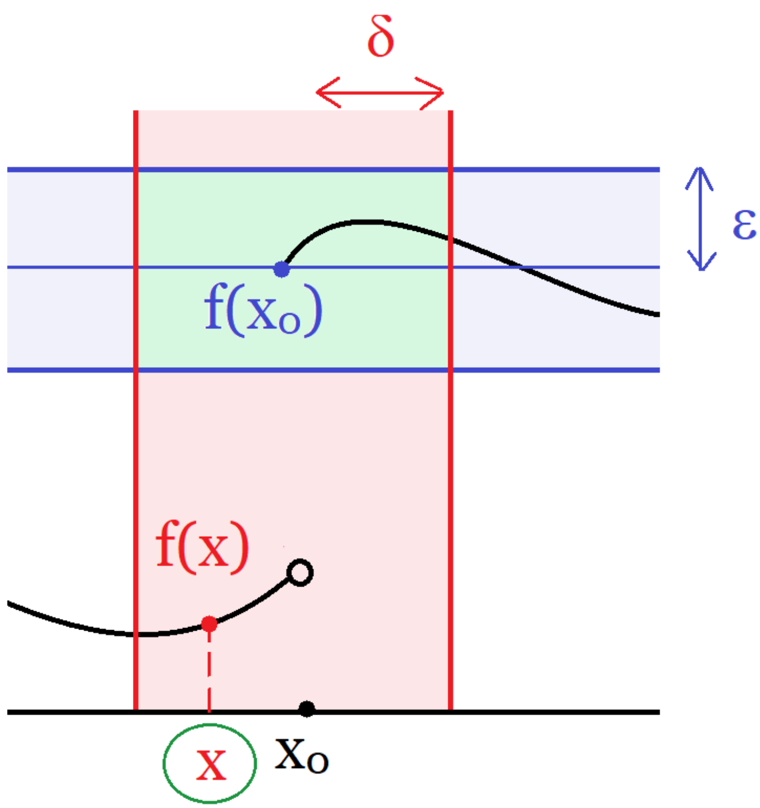
The $\epsilon - \delta$ Way:

What does it mean for a function to **not** be continuous? For this, let's negate the statement

For all $\epsilon > 0$ there is $\delta > 0$ such that for all x, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$

Definition :

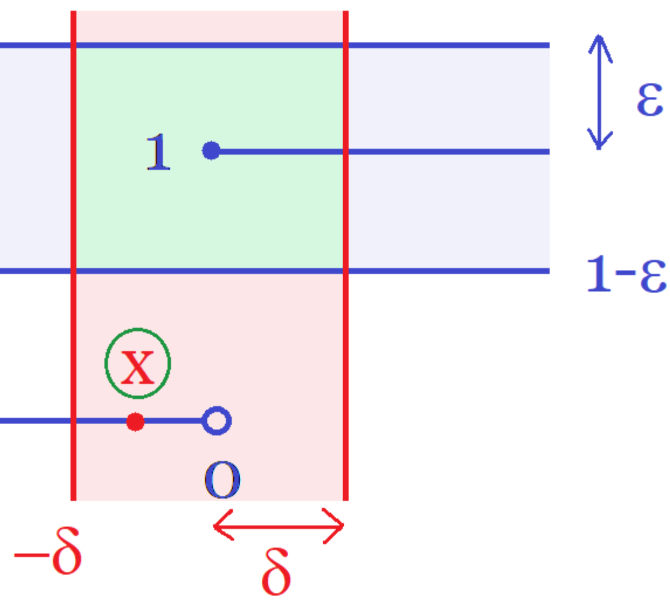
f is **not continuous** at x_0 if there is $\epsilon > 0$ such that for all $\delta > 0$ there is x such that $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \epsilon$.



Interpretation: There is an error ϵ, such that, no matter how δ-close we get to x_0, there is always a counterexample x such that $f(x)$ is outside of the good region with $< \epsilon$, as in the following picture (the good region is in blue/green)

In this case, with $x_0 = 0$ and $f(x_0) = f(0) = 1$, this becomes:

There is an $\epsilon > 0$ such that for all $\delta > 0$, there is x such that $|x - 0| = |x| < \delta$, but $|f(x) - f(0)| = |f(x) - 1| \geq \epsilon$



The figure above suggests the following:

Let $\epsilon = \frac{1}{2}$ (any ϵ smaller than 1 works), then given $\delta > 0$, let x be any *negative* number such that $|x| < \delta$ (for example, $x = -\frac{\delta}{2}$ works), then $|x| < \delta$ but since $x < 0$,

$$|f(x) - f(0)| = |0 - 1| = 1 \geq \frac{1}{2} = \epsilon \checkmark \quad \square$$

6. Optional: Alternating Series Test Proof

Alternating Series Test:

If $a_n \geq 0$, is decreasing, and $a_n \geq 0$ then $\sum(-1)^n a_n$ converges

Example 3:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

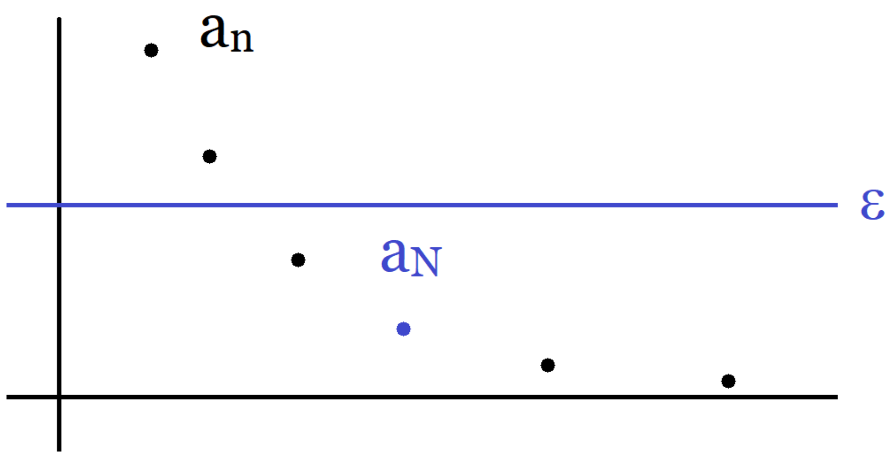
converges because $a_n = \frac{1}{n} \geq 0$ and is decreasing

Proof: For this, we need to use the Cauchy Criterion:

Recall: Cauchy Criterion

$\sum a_n$ converges if and only if for all $\epsilon > 0$ there is N such that if $n \geq m > N$, then $\left|\sum_{k=m}^{n} a_n\right| < \epsilon$

Let $\epsilon > 0$ be given. Since $a_n \to 0$ there some N such that $a_N < \epsilon$



With N as above, if $n \geq m > N$, let's show the following:

Claim:

$$\left|\sum_{k=m}^{n}(-1)^k a_k\right| \leq a_N$$

Then we would be done because we'd have

$$\left|\sum_{k=m}^{n}(-1)^k a_k\right| \leq a_N < \epsilon$$

Therefore $\sum(-1)^n a_n$ converges by the Cauchy criterion.

Proof of Claim: Notice:

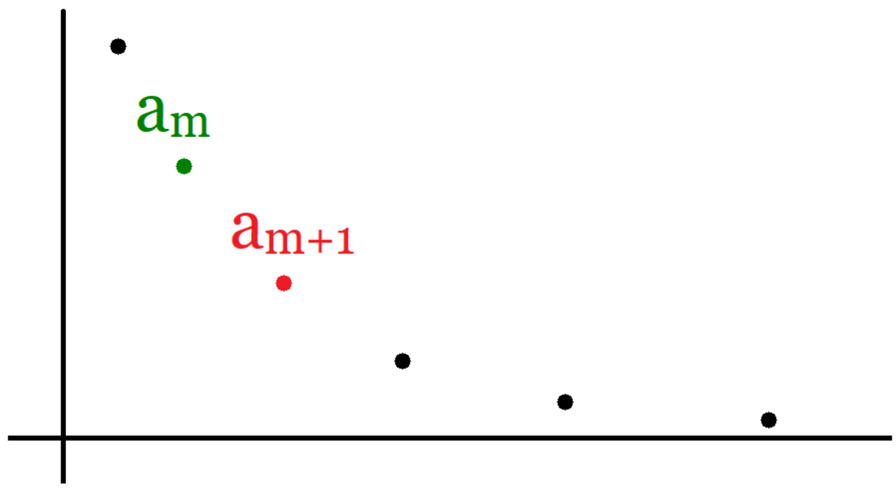
$$\begin{aligned}
\left|\sum_{k=m}^{n}(-1)^k a_k\right| &= \left|(-1)^m a_m + (-1)^{m+1} a_{m+1} + \cdots + (-1)^n a_n\right| \\
&= \left|\underbrace{(-1)^m}_{\pm 1}\left(a_m - a_{m+1} + \cdots + (-1)^{n-m} a_n\right)\right| \\
&= \left|a_m - a_{m+1} + \ldots (-1)^{n-m} a_n\right|
\end{aligned}$$

Case 1: $n - m$ is odd

Then $(-1)^{n-m} = -1$ (so our sum looks something like $a_3 - a_4 + a_5 - a_6$), and so

$$\begin{aligned}
\left|a_m - a_{m+1} + \ldots (-1)^{n-m} a_n\right| &= \left|a_m - a_{m+1} + \cdots - a_n\right| \\
&= \left|\underbrace{(a_m - a_{m+1})}_{\geq 0} + \cdots + \underbrace{(a_{n-1} - a_n)}_{\geq 0}\right| \\
&= a_m - a_{m+1} + \cdots + a_{n-1} - a_n
\end{aligned}$$

(Here we used the fact that (a_n) is decreasing, and so, for example, $a_m \geq a_{m+1}$, hence $a_m - a_{m+1} \geq 0$)



On the other hand, we can write this as (Analogy: Think $a_3 - a_4 + a_5 - a_6 = a_3 - (a_4 - a_5) - a_6$)

$$
\begin{aligned}
a_m - a_{m+1} + \cdots - a_n =& a_m - a_{m+1} + a_{m+2} + \cdots - a_{n-2} + a_{n-1} - a_n \\
=& a_m - \Bigg(\underbrace{a_{m+1} - a_{m+2}}_{\geq 0}\Bigg) - \cdots - \Bigg(\underbrace{a_{n-2} - a_{n-1}}_{\geq 0}\Bigg) - a_n \\
\leq& a_m - \underbrace{a_n}_{\geq 0} \\
\leq& a_m \\
\leq& a_N \checkmark
\end{aligned}
$$

(the last inequality follows because $m > N$ and (a_n) is decreasing)

$$
\text{Hence } \left| \sum_{k=m}^{n} (-1)^k a_k \right| = |a_m - a_{m+1} + \cdots - a_n| \leq a_N \checkmark
$$

And we are done in the case where $n-m$ is odd

Case 2: $n-m$ is even

Then $(-1)^{n-m}=1$ (so our sum looks something like $a_3-a_4+a_5-a_6+a_7=(a_3-a_4)+(a_5-a_6)+a_7$)

Therefore:

$$\begin{aligned}\left|a_m-a_{m+1}+\dots(-1)^{n-m}a_n\right| &= |a_m-a_{m+1}+\cdots+a_n| \\ &= \left|\underbrace{(a_m-a_{m+1})}_{\geq 0}+\cdots+\underbrace{(a_{n-2}-a_{n-1})}_{\geq 0}+\underbrace{a_n}_{\geq 0}\right| \\ &= a_m-a_{m+1}+\cdots+a_n\end{aligned}$$

On the other hand, we can write this as (Analogy: Think $a_3-a_4+a_5-a_6+a_7=a_3-(a_4-a_5)-(a_6-a_7)$)

$$\begin{aligned}a_m-a_{m+1}+\cdots+a_n &= a_m-a_{m+1}+a_{m+2}+\cdots-a_{n-1}+a_n \\ &= a_m-\left(\underbrace{a_{m+1}-a_{m+2}}_{\geq 0}\right)-\cdots-\left(\underbrace{a_{n-1}-a_n}_{\geq 0}\right) \\ &\leq a_m \\ &\leq a_N\end{aligned}$$

$$\text{Hence }\left|\sum_{k=m}^{n}(-1)^k a_k\right| = |a_m-a_{m+1}+\cdots+a_n| \leq a_N \checkmark$$

And we are done in this case as well $\square$