LECTURE 15: THE GRADIENT AND APPLICATIONS

Today: Some fun applications of partial derivatives!

1. The Gradient Vector

Now that we've calculated things like f_x or f_y, you might ask: What if we put them together? This is the idea behind gradient vectors.

Example 1:

Calculate ∇f, where $f(x,y) = x^2y + xy^2$

Definition

$$\nabla f = \langle f_x, f_y \rangle$$

(Literally just calculate f_x and f_y and put them together)

$$f_x = \left(x^2y + xy^2\right)_x = 2xy + y^2$$
$$f_y = \left(x^2y + xy^2\right)_y = x^2 + 2xy$$

$$\text{Therefore} \quad \nabla f = \left\langle 2xy + y^2, x^2 + 2xy \right\rangle$$

Note: ∇f is a **vector**. Think of it as the "derivative" of f, tells you in which direction f is moving.

Example 2:

Calculate ∇f, where $f(x,y,z) = x\sin(yz) + \ln(yz) + x$

Date: Friday, October 1, 2021.

Definition

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$f_x = (x\sin(yz) + \ln(yz) + x)_x = \sin(yz) + 1$$
$$f_y = (x\sin(yz) + \ln(yz) + x)_y = x\cos(yz)(z) + \frac{1}{yz}(z) = xz\cos(yz) + \frac{1}{y}$$
$$f_z = (x\sin(yz) + \ln(yz) + x)_z = x\cos(yz)(y) + \frac{1}{yz}(y) = xy\cos(yz) + \frac{1}{z}$$

$$\nabla f = \langle f_x, f_y, f_z \rangle = \left\langle \sin(yz) + 1, xz\cos(yz) + \frac{1}{y}, xy\cos(yz) + \frac{1}{z} \right\rangle$$

Today's Goal:

What is ∇f useful for?

2. Gradient and Level Curves

Example 3: (Motivation)

(a) Calculate $\nabla f(1,2)$ where $f(x,y) = x^2 + y^2$

$$\nabla f = \langle f_x, f_y \rangle = \left\langle (x^2+y^2)_x, (x^2+y^2)_y \right\rangle = \langle 2x, 2y \rangle$$

Now let $x = 1$ and $y = 2$ to get

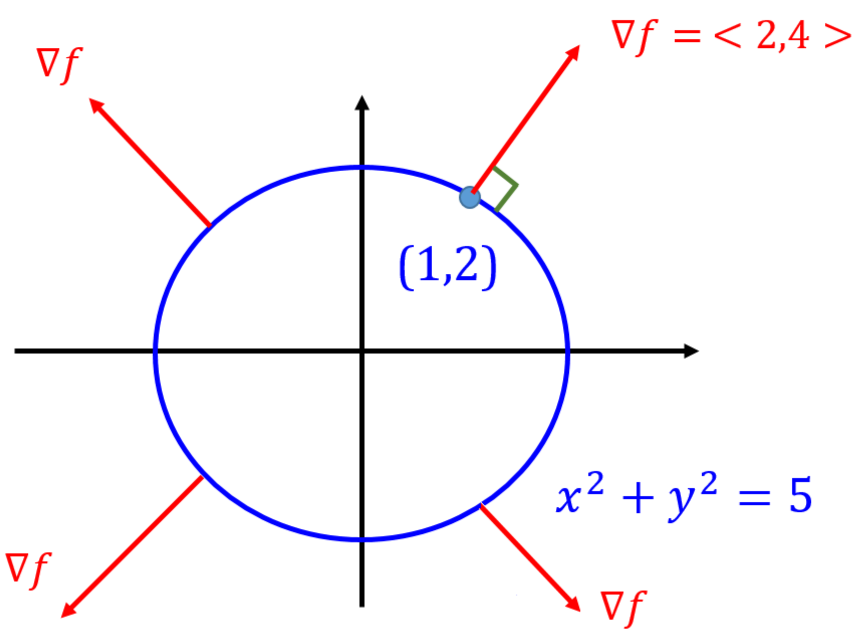
$$\nabla f(1,2) = \langle 2(1), 2(2) \rangle = \langle 2, 4 \rangle$$

(b) Draw the level curve $f(x,y) = 5$, as well as $\nabla f(1,2)$

(Recall that level curves are curves of the form $z =$ some number, like heights of mountains)

$$f(x, y) = 5 \Rightarrow x^2 + y^2 = 5 \text{ (Circle of radius } \sqrt{5}\text{)}$$

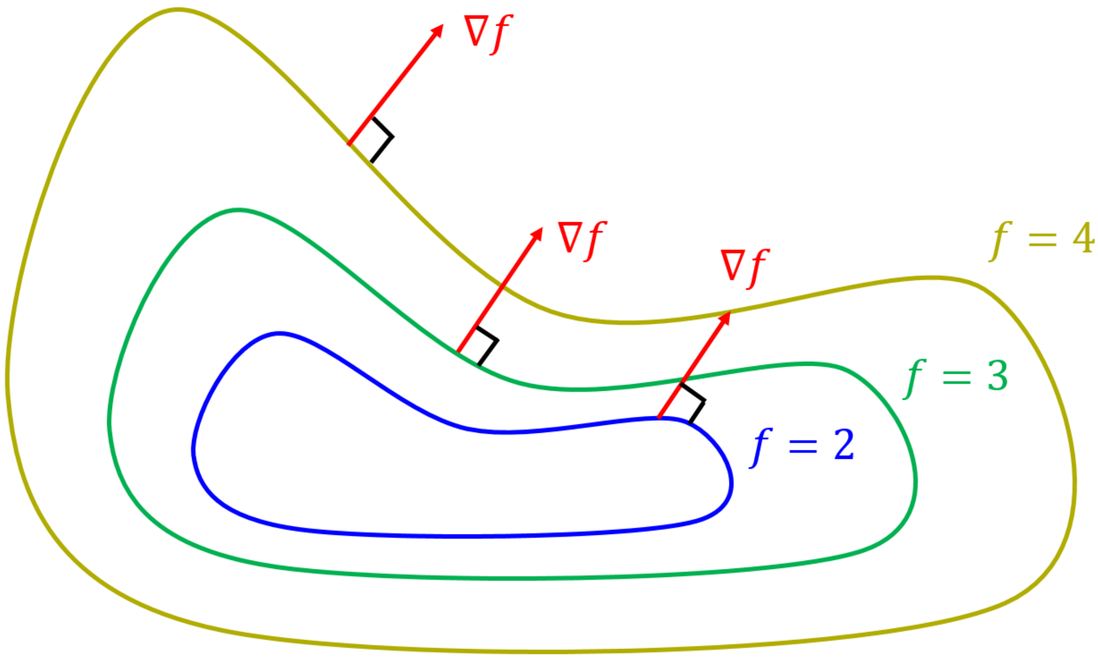
$$\nabla f(1, 2) = \langle 2, 4 \rangle$$



Notice here that ∇f is perpendicular to the circle $x^2 + y^2 = 5$, and in fact this is always true:

Ultra Important Fact:

∇f is always perpendicular to level curves (or surfaces) of f



Note: This kind of makes sense: ∇f tells you in which direction f is moving, but on level curves like $f = 2$, f is constant, so ∇f should move away from the level curve as much as possible, so it's *perpendicular* to it.

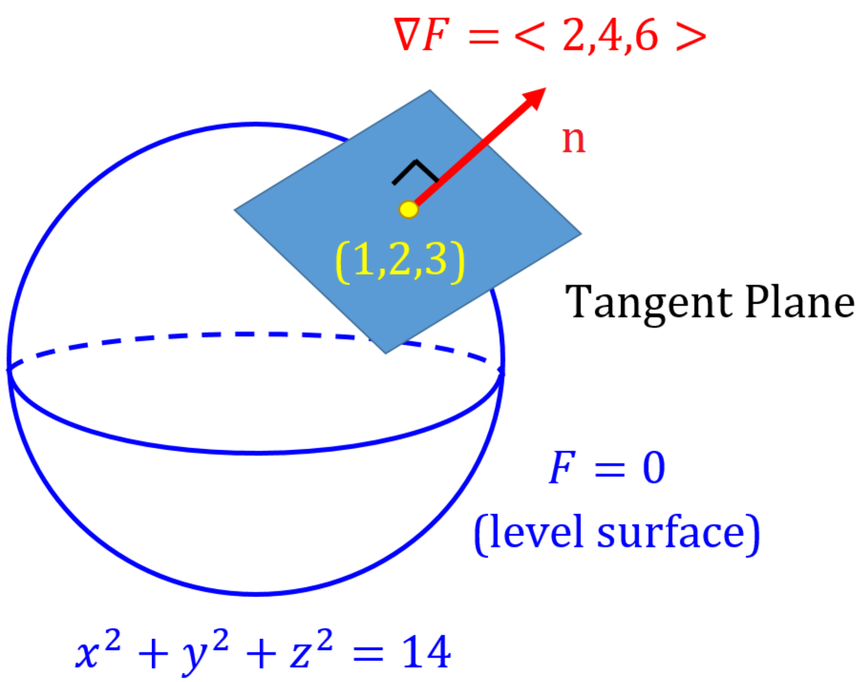
3. Tangent Planes to Surfaces

Let's explore the above fact a bit further and see why this is useful.

Example 4:

Find the equation of the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 14 \text{ at } (1, 2, 3)$$



This is interesting, because even though $x^2+y^2+z^2 = 14$ is not a function, it still has a tangent plane, and we can still calculate its equation.

STEP 1: Rewrite $x^2 + y^2 + z^2 = 14$ as

$$\underbrace{x^2 + y^2 + z^2 - 14}_{F(x,y,z)} = 0 \Rightarrow F(x, y, z) = 0 \text{ (Level Surface)}$$

STEP 2: To find the equation of a plane, we need:

(1) **Point:** $(1, 2, 3)$

(2) **Normal Vector:** Need a vector perpendicular to the tangent plane. But by the picture above, we get:

Fact:

$$\mathbf{n} = \nabla F(1,2,3)$$

Here, using $F = x^2 + y^2 + z^2 - 14$, we get:

$$\nabla F = \langle F_x, F_y, F_z \rangle = \langle 2x, 2y, 2z \rangle$$

$$\mathbf{n} = \nabla F(1,2,3) = \langle 2(1), 2(2), 2(3) \rangle = \langle 2, 4, 6 \rangle$$

(3) **Equation:** Since Point $= (1,2,3)$ and $\mathbf{n} = \langle 2,4,6 \rangle$, we get:

$$2(x-1) + 4(y-2) + 6(z-3) = 0$$

Summary:

The equation of the tangent plane to $F(x,y,z) = 0$ at $(1,2,3)$ is

$$F_x(1,2,3)(x-1) + F_y(1,2,3)(y-2) + F_z(1,2,3)(z-3) = 0$$

Example 5: (Good Quiz/Exam Question)

Find the equation of the tangent plane to the surface

$$xyz = z^3 \text{ at } (1,4,2)$$

(1) $F(x,y,z) = xyz - z^3$

(2) **Point:** $(1,4,2)$

(3) **Normal Vector:**

$$\nabla F = \langle F_x, F_y, F_z \rangle = \left\langle (xyz - z^3)_x, (xyz - z^3)_y, (xyz - z^3)_z \right\rangle = \left\langle yz, xz, xy - 3z^2 \right\rangle$$

$$\mathbf{n} = \nabla F(1,4,2) = \left\langle (4)(2), (1)(2), (1)(4) - 3(2)^2 \right\rangle = \langle 8, 2, -8 \rangle$$

(4) **Equation:**

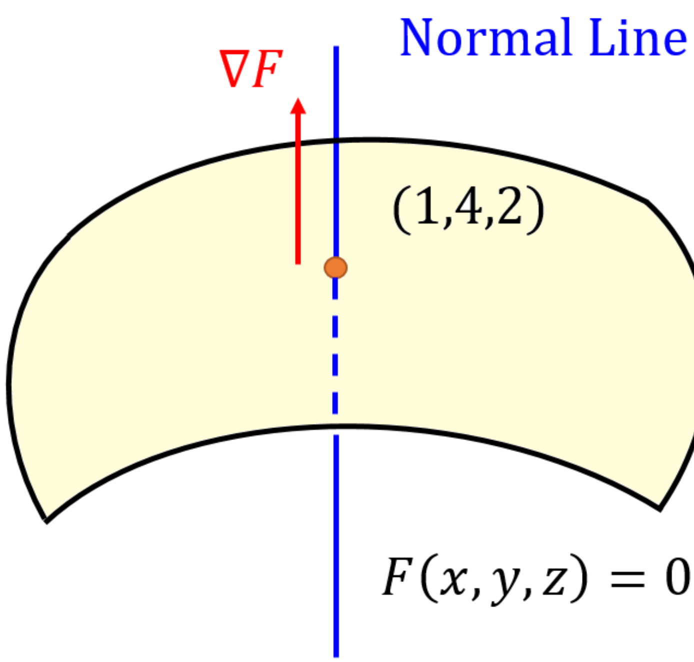
$$8(x-1) + 2(y-4) - 8(z-2) = 0$$

Related to this is the concept of a normal line, which is just a line perpendicular to your surface.

Example 6: (Normal Line, extra practice)

Find the parametric equations of the normal line to the surface

$$xyz = z^3 \text{ at } (1,4,2)$$



(1) $F(x,y,z) = xyz - z^3$

(2) **Point:** $(1, 4, 2)$

(3) **Direction Vector:** Here the direction vector is the normal vector to the tangent plane, which is $\nabla F(1, 4, 2)$:

$$\text{Direction Vector: } \nabla F(1, 4, 2) = \langle 8, 2, -8 \rangle$$

(4) **Parametric Equations:**

$$\begin{cases} x(t) = 1 + 8t \\ y(t) = 4 + 2t \\ z(t) = 2 - 8t \end{cases}$$

4. Directional Derivatives

As another application of gradients, here we'll take the derivative of f in any direction.

Recall: f_x is the derivative of f in the $x-$direction (similar for f_y).

But what about other directions like $\langle 1, 1 \rangle$ or $\langle 2, 3 \rangle$? Turns out that we can use ∇f to calculate derivatives in other directions as well!

Important:

In what follows, $\mathbf{u}$ **MUST** be a unit vector (= Length 1)

Example 7:

(a) Find the **directional derivative** $D_{\mathbf{u}} f$ of $f(x, y) = 3x^2 + 4y^2 + 1$ in the direction $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

Definition:

If $\mathbf{u}$ is a unit vector, then $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$

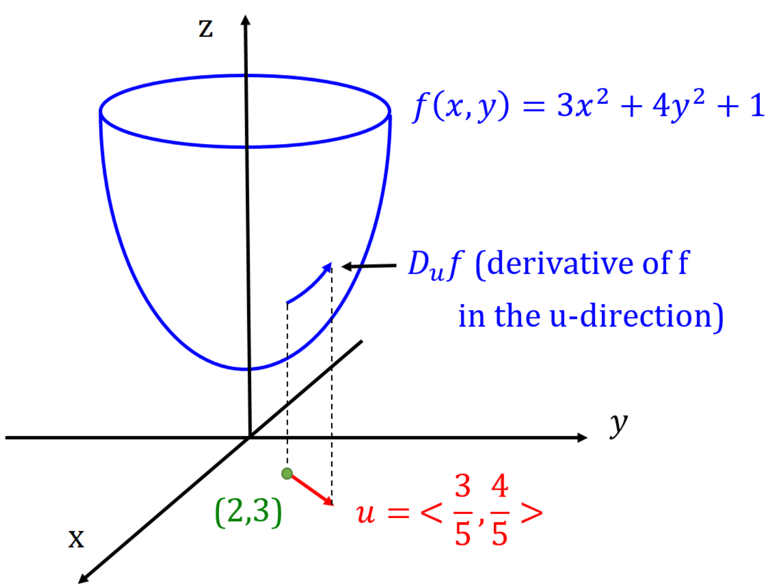
$$\nabla f = \left\langle \left(3x^2+4y^2+1\right)_x, \left(3x^2+4y^2+1\right)_y \right\rangle = \langle 6x, 8y\rangle$$

$$D_{\mathbf{u}}f = (\nabla f)\cdot\mathbf{u} = \langle 6x, 8y\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle = (6x)\left(\frac{3}{5}\right) + (8y)\left(\frac{4}{5}\right) = \left(\frac{18}{5}\right)x + \left(\frac{32}{5}\right)y$$

(b) Find $D_{\mathbf{u}}f$ at $(2,3)$

$$D_{\mathbf{u}}f(2,3) = \left(\frac{18}{5}\right)\times 2 + \left(\frac{32}{5}\right)\times 3 = \frac{36+96}{5} = \frac{132}{5} = 26.4$$

Interpretation: $D_{\mathbf{u}}f$ is the derivative (= rate of change) of f in the direction of $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5}\rangle$



It's important to remember that **u must** be a unit vector

Example 8:

Find $D_{\mathbf{u}}f(1,2)$, where $f(x,y) = x^3y$ and $\mathbf{u} = \langle 1, 1\rangle$

(1)

$$\nabla f = \left\langle \left(x^3y\right)_x, \left(x^3y\right)_y \right\rangle = \left\langle 3x^2y, x^3 \right\rangle$$

$$\nabla f(1,2) = \left\langle 3(1)^2(2), 1^3 \right\rangle = \langle 6, 1\rangle$$

(2) ⚠ **u** is **not** a unit vector, so **cannot** use $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$

Solution: Turn **u** into a unit vector

Recall:

$$\mathbf{u}' = \frac{\mathbf{u}}{\|\mathbf{u}\|} \quad \text{always has length 1}$$

$$\text{Here: } \mathbf{u}' = \frac{\langle 1, 1\rangle}{\sqrt{1^2+1^2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

(3)

$$D_{\mathbf{u}}f(1,2) = \nabla f(1,2) \cdot \mathbf{u}' = \langle 6, 1\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{6}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$

Note: Sometimes you are asked to find the directional derivative with an angle $\theta = \frac{\pi}{3}$ In that case you just use

$$\mathbf{u} = \langle \cos(\theta), \sin(\theta) \rangle = \left\langle \cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right) \right\rangle = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

5. Maximal Increase

As a last application of gradients, you may ask: In which direction does f increase the fastest?

(Think like Mario Kart: Which direction do you have to drive in to get first to the finish line?)

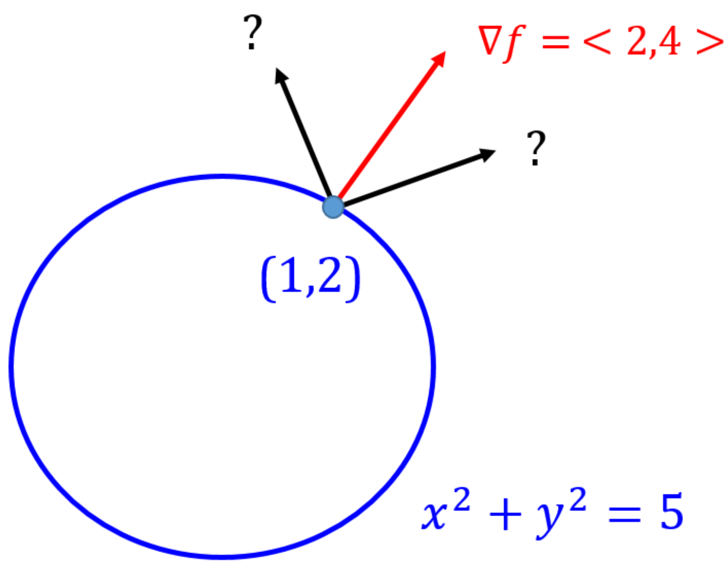
Example 9:

(a) Consider $f(x, y) = x^2 + y^2$ at $(1, 2)$. In which direction does f increase the fastest?

Previously, we found $\nabla f = \langle 2x, 2y \rangle$ and $\nabla f(1, 2) = \langle 2, 4 \rangle$

Fact:

The fastest increase is in the direction of $\nabla f = \langle 2, 4 \rangle$



And the fastest decrease is in the direction of $-\nabla f = \langle -2, -4 \rangle$

Intuitively: This makes sense, because on the level curve $x^2 + y^2 = 5$, f is constant, so to increase fastest, you need to point away from it.

Analogy: In fact, if you're on a mountain (or cliff) and your friend says to jump from it to the ground, you usually jump perpendicular to the mountain, and not tangential to it.

(b) What is the biggest rate of increase of f?

Fact:

The biggest increase is $\|\nabla f\|$

Hence here it's $\|\langle 2, 4 \rangle\| = \sqrt{2^2 + 4^2} = \sqrt{20}$.

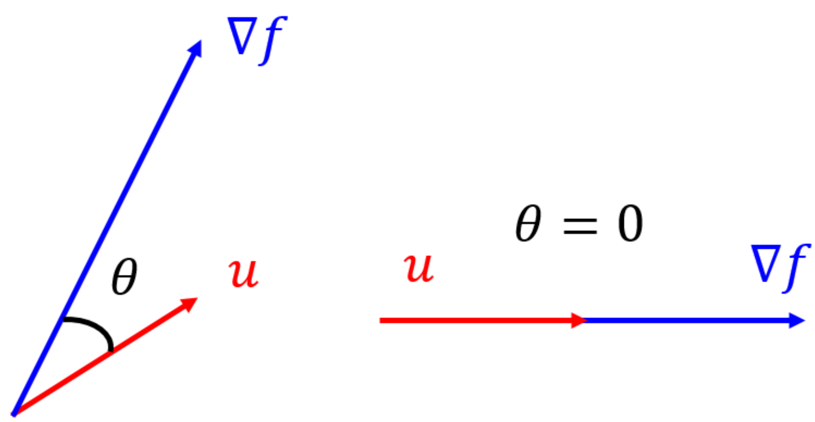
In summary: The gradient ∇f helps us find tangent planes, normal lines, directional derivatives, and greatest rate of increase, so it's pretty powerful!

Optional: Proof of Facts:

We need to figure out in which direction $\mathbf{u}$ is $D_{\mathbf{u}} f$ the greatest? But using the definition of $D_{\mathbf{u}} f$ and the angle formula, we get

$$D_{\mathbf{u}} f = (\nabla f) \cdot \mathbf{u} = \|\nabla f\| \underbrace{\|\mathbf{u}\|}_{1} \cos(\theta) = \|\nabla f\| \cos(\theta)$$

Where θ is the angle between $\mathbf{u}$ and ∇f



Now $\|\nabla f\| \cos(\theta)$ is the greatest when $\theta = 0$, which means that $\mathbf{u}$ and ∇f have the same direction (see figure above). Hence f increases fastest in the direction of $\mathbf{u}$.

And finally, the greatest rate of increase is when $\theta = 0$, that is

$$\|\nabla f\| \cos(0) = \|\nabla f\| \quad \square$$