LECTURE 15: THE INVERSE FUNCTION THEOREM

Welcome to the first of two major theorems about derivatives: The Inverse Function Theorem

1. Motivation

Goal: If $y = f(x)$, when can we solve for x in terms of y? That is, when can we write $x = g(y)$ where g is a **smooth** function?

Example 1: If $f(x) = x^3$ then $g(y) = y^{\frac{1}{3}}$. Notice g is differentiable **except** at 0, and 0 is *precisely* the point where $f'(x) = 0$

Example 2: If $f(x) = x^2$ then we can't find a *global* inverse (valid for all x) since f isn't one-to-one, but our hope is to do this locally, around a point. Once again there is no inverse when $f'(x) = 0$.

In short, we would like to say "As long as $f'(x) \neq 0$, we can solve for x in terms of y, at least locally"

2. Inverse Function Theorem (Part 1)

Definition: f is C^1 if f is differentiable and f' is continuous.

Definition: f is **invertible** if there is g such that $f(g(x)) = g(f(x)) = x$ for all x.

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This is equivalent to f being one-to-one and onto.

Definition: A matrix A is **invertible** if there is A^{-1} such that $AA^{-1} = A^{-1}A = I$. Equivalently, $\det(A) \neq 0$

Inverse Function Theorem 1:

Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. If $\det(f'(a)) \neq 0$ for some a, there is an open neighborhood U of a and an open neighborhood V of $f(a)$ such that $f : U \to V$ is invertible.

This theorem is incredibly powerful. It says that f **inherits** invertibility from its derivative f'. So if f' is invertible (as a linear transformation), then f is invertible (as a function), at least locally.

This is one of the theorems that illustrates how f' "controls" f.

Intuitively, this makes sense in terms of linear approximations. If $r(h) = 0$, then we get

$$f(x + h) = f(x) + f'(x)h$$

So if $f'(x)$ is invertible, then $f(x + h) \neq f(x)$ at least for small h

The proof is a surprising application of:

Recall: [Banach Fixed point Theorem]

Let (X, d) be a complete metric space and $\phi : X \to X$ a contraction, then ϕ has a unique fixed point.

Proof of Inverse Function Theorem 1:

PART 1: Find U and show f is one-to-one on U

STEP 1: Let $A = f'(a)$ and let λ (small) be TBA

Intuitively: Since f' is continuous at a, if x is close to a then $f'(x)$ is close to $f'(a) = A$

More precisely, there is $r > 0$ small such that for all $x \in B(a, r)$,

$$\|f'(x) - A\| < \lambda$$

Let $\boxed{U = B(a, r)}$ (open)

STEP 2: Fix $y \in \mathbb{R}^n$ and define the following function $\phi : \mathbb{R}^n \to \mathbb{R}^n$

$$\phi(x) = x + A^{-1}(y - f(x))$$

Notice that $f(x) = y \Leftrightarrow \phi(x) = x$ (since the terms in parentheses are 0)

Our ultimate goal is to apply the Banach Fixed Point theorem to ϕ

STEP 3: Claim: ϕ is a contraction

Why? By the Chain Rule, we have

$$\phi'(x) = I + A^{-1}\left(-f'(x)\right) = A^{-1}A - A^{-1}f'(x) = A^{-1}\left(A - f'(x)\right)$$

$$\text{Hence } \|\phi'(x)\| \leq \left\|A^{-1}\right\| \|A - f'(x)\| \overset{\textbf{STEP 1}}{<} \left\|A^{-1}\right\| \lambda$$

Therefore, if we *choose* λ such that $\left\|A^{-1}\right\| \lambda < \frac{1}{2}$ then we get

$$\|\phi'(x)\| < \frac{1}{2}$$

By the version of the Mean Value Theorem from last time, we get

$$|\phi(x_1) - \phi(x_2)| \leq \frac{1}{2} |x_1 - x_2|$$

Hence ϕ is a contraction

STEP 4: f is one-to-one on U

Suppose $f(x_1) = f(x_2) = y$ then $\phi(x_1) = x_1$ and $\phi(x_2) = x_2$ and

$$|x_1 - x_2| \leq \frac{1}{2} |x_1 - x_2|$$

Which implies $x_1 = x_2$ so f is one-to-one

So if you define $\boxed{V = f(U)}$ then $f : U \to V$ is invertible.

The only thing we need to show is that:

PART 2: V is open

This is where we'll need the full version of the Banach Fixed Point Theorem. That is, we will need to find the complete metric space X and show that $\phi : X \to X$

STEP 1: Let $y_0 \in V = f(U)$. Then $y_0 = f(x_0)$ for some $x_0 \in U$

We need to show that $y \in f(U)$ for all y close enough to y_0

Fix y such that $|y - y_0| <$ TBA (small)

Let $\boxed{B = B(x_0, r)}$ where r is so small that $\overline{B} \subseteq U$

STEP 2: Claim: $\phi : \overline{B} \to \overline{B}$

Why? First of all, notice

$$|\phi(x_0) - x_0| = \left|\cancel{x_0} - A^{-1}(y - f(x_0)) - \cancel{x_0}\right| = \left|A^{-1}(y - y_0)\right| \leq \left\|A^{-1}\right\| |y - y_0| < \frac{1}{2\lambda}|y - y_0|$$

But now if you *choose* y such that $|y - y_0| < \lambda r$ then the above becomes

$$|\phi(x_0) - x_0| < \frac{1}{2\lambda}(\lambda r) = \frac{r}{2}$$

Therefore if $x \in \overline{B}$ then we get by the contraction property

$$|\phi(x) - x_0| \leq |\phi(x) - \phi(x_0)| + |\phi(x_0) - x_0| < \frac{1}{2}\underbrace{|x - x_0|}_{\leq r} + \frac{r}{2} \leq \frac{r}{2} + \frac{r}{2} = r$$

(Here we used $|x - x_0| \leq r$ since $x \in \overline{B}$)

Hence $\phi(x) \in \overline{B(x_0, r)} =: \overline{B}$ ✓

STEP 3: Hence $\phi : \overline{B} \to \overline{B}$ is a contraction. And since $\overline{B}$ is a closed subset of the complete metric space $\mathbb{R}^n$, $\overline{B}$ is complete.

Therefore by the Banach fixed point theorem, ϕ has a unique fixed point x.

For this x, by definition of ϕ, we have $f(x) = y$

STEP 4: Conclusion:

We have shown that if $y_0 \in V$ and if $y \in B(y_0, \lambda r)$ then $y = f(x)$ for some $x \in U$, that is $y \in f(U) = V$. Therefore $B(y_0, \lambda r) \subseteq V$ and thus we have shown that V is open □

Corollary: [Open Mapping Theorem] Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and $f'(x)$ is invertible for all x. If U is an open subset of $\mathbb{R}^n$ then $f(U)$ is open as well

So f maps open sets to open sets.

This is interesting because in topology, f is continuous $\Leftrightarrow$ $f^{-1}(U)$ is open whenever U is open, so this is saying that f^{-1} is continuous.

3. Inverse Function Theorem (Part 2)

Not only does the inverse g exist, but it's actually differentiable!

Motivation: $(n = 1)$ If $f(g(x)) = x$ then differentiating this, we get

$$f'(g(x))g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))}$$

This was used in Calculus to get the derivatives of $\ln(x)$ or $\sin^{-1}(x)$ for example.

And in fact the same thing is true in higher dimensions.

Inverse Function Theorem 2

If $g : V \to U$ is the inverse of f, defined by $g(f(x)) = x$

$$\text{Then } g \in C^1(V) \text{ and } g'(x) = (f'(g(x)))^{-1}$$

Proof of Inverse Function Theorem 2:

STEP 1: Fix $y \in V$ and let k small enough so that $y + k \in V$ (we will ultimately let $k \to 0$)

By definition of $V = f(U)$ there is x such that $y = f(x)$ and z such that $y + k = f(z)$. Note that you can write $z = x + \underbrace{(z - x)}_{h} = x + h$

So $\boxed{y = f(x)}$ and $\boxed{y + k = f(x + h)}$ for some h

STEP 2: We would like to compare $|h|$ with $|k|$

Claim: $|h| \leq \frac{1}{\lambda} |k|$

This implies in particular that if $k \to 0$ then $h \to 0$.

Why? Notice

$$\begin{aligned}\phi(x+h) - \phi(x) =& \left(\cancel{x} + h + A^{-1}\left(\cancel{y} - f(x+h)\right)\right) - \left(\cancel{x} + A^{-1}\left(\cancel{y} - f(x)\right)\right) \\ =& h - A^{-1}\left(f(x+h) - f(x)\right) \\ =& h - A^{-1}\left(y + k - y\right) \\ =& h - A^{-1}k\end{aligned}$$

$$\text{Hence } \left|h - A^{-1}k\right| = |\phi(x+h) - \phi(x)| \leq \frac{1}{2}|x + h - x| = \frac{1}{2}|h|$$

And by the Reverse Triangle inequality, we have

$$\left|A^{-1}k\right| = \left|A^{-1}k - h - (-h)\right| \geq \left|\left|A^{-1}k - h\right| - |-h|\right| \geq \left|\frac{1}{2}|h| - |h|\right| = \left|-\frac{1}{2}|h|\right| = \frac{|h|}{2}$$

$$\text{Hence } |h| \leq 2\left|A^{-1}k\right| \leq \underbrace{2\left\|A^{-1}\right\|}_{\frac{1}{\lambda}}|k| = \frac{1}{\lambda}|k| \checkmark$$

STEP 3:

Recall: If A is invertible and $\|B-A\| < \frac{1}{\|A^{-1}\|}$ then B is invertible

Since $A = f'(a)$ is invertible (by assumption), the fact that

$$\|f'(x) - A\| < \lambda < \frac{1}{2\|A^{-1}\|} < \frac{1}{\|A^{-1}\|}$$

Implies that $f'(x)$ is invertible for all $x \in U$

Let $\boxed{T = (f'(x))^{-1}}$

Claim: $g'(y) = T$

Then we would be done because then

$$g'(y) = T = (f'(x))^{-1} = (f'(g(y)))^{-1} \checkmark$$

STEP 4: Proof of Claim:

$$\begin{aligned}
\underbrace{f(x+h)}_{y+k} - \underbrace{f(x)}_{y} - f'(x)h =& k - f'(x)h \\
&= (f'(x))\left((f'(x))^{-1}k - h\right) \\
&= (f'(x))(Tk - h) \\
&= (f'(x))\left(Tk - (\underbrace{x+h}_{g(y+k)} - \underbrace{x}_{g(y)})\right) \\
&= (f'(x))(Tk - g(y+k) + g(y)) \\
&= -f'(x)(g(y+k) - g(y) - Tk)
\end{aligned}$$

Multiplying both sides by $-(f'(x))^{-1} = -T$ it then follows that

$$g(y+k) - g(y) - Tk = -T(f(x+h) - f(x) - f'(x)h)$$

$$\frac{|g(y+k)-g(y)-Tk|}{|k|} \leq \frac{\|T\|\,|f(x+h)-f(x)-f'(x)h|}{|k|}$$
$$= \|T\| \left(\frac{|h|}{|k|}\right) \frac{|f(x+h)-f(x)-f'(x)h|}{|h|}$$
$$\overset{\textbf{STEP 1}}{\leq} \|T\| \left(\frac{1}{\lambda}\right) \frac{|f(x+h)-f(x)-f'(x)h|}{|h|}$$

Now let $k \to 0$.

Then, by **STEP 1** we have $h \to 0$ and so the right-hand-side of goes to 0 by definition of $f'(x)$, which forces the left-hand-side goes to 0.

Therefore in fact $g'(y) = T$ ✓

STEP 5: $g \in C^1$

Since g is differentiable, g is continuous, therefore $f'(g(y))$ is continuous, and so is $(f'(g(y)))^{-1} = g'(y)$ since the mapping $A \to A^{-1}$ is continuous □

Definition: f is a C^1 **diffeomorphism** if f is C^1, one-to-one, onto, and f^{-1} is C^1

Corollary: If $\det(f'(a)) \neq 0$, for some a, then f is locally a C^1 diffeomorphism.