LECTURE 16: CONTINUOUS FUNCTIONS (II)

1. Sequential Vs. $\epsilon - \delta$ Continuity

Video: Equivalent Definitions

Let's show that the two definitions of continuity are equivalent:

Definition 1:

f is **continuous at** x_0 if, whenever (x_n) is a sequence that converges to x_0, then $f(x_n)$ converges to $f(x_0)$

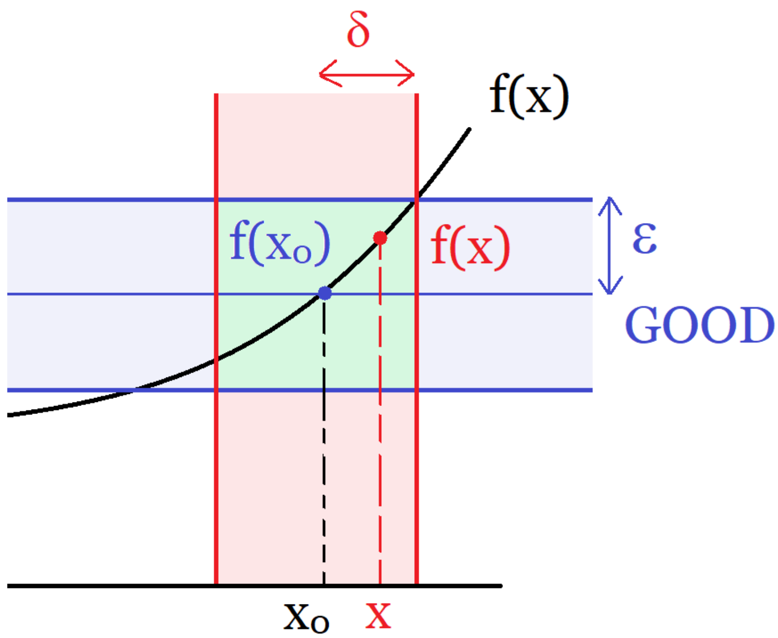
Definition 2:

f is **continuous** at x_0 if for all $\epsilon > 0$ there is $\delta > 0$ such that, for all x, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$

Fact:

The two definitions are equivalent

Date: Thursday, October 21, 2021.



Proof: (Definition 2 $\Rightarrow$ Definition 1)

Suppose the $\epsilon - \delta$ definition holds, and let (x_n) be a sequence that converges to x_0.

Goal: Show $f(x_n)$ converges to $f(x_0)$.

Let $\epsilon > 0$ be given

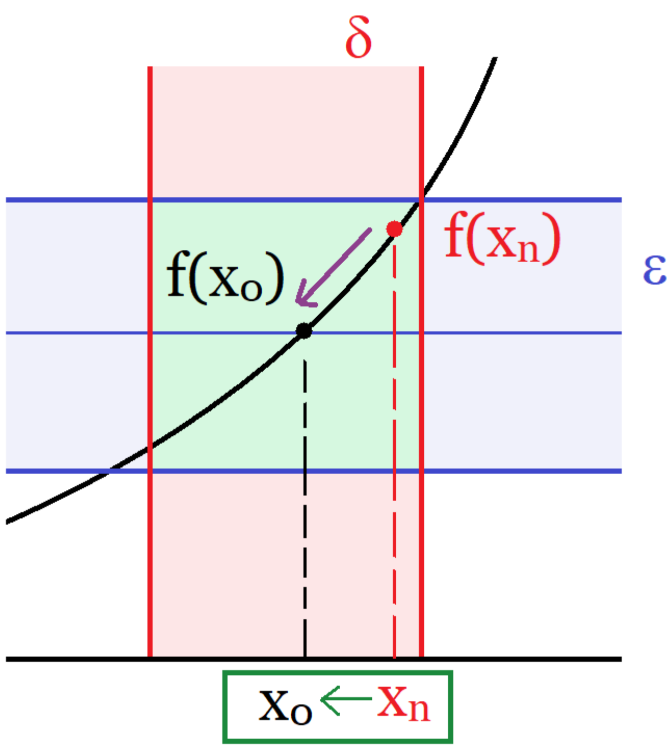
Then, by $\epsilon - \delta$, there is $\delta > 0$ such that for all x, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$ (this is just $\epsilon - \delta$)

However, since $x_n \to x_0$, by definition of the limit of sequences (but with δ instead of ϵ) there is N such that if $n > N$, then $|x_n - x_0| < \delta$

But since $|x_n - x_0| < \delta$, by $\epsilon - \delta$, we have $|f(x_n) - f(x_0)| < \epsilon$.

So for all $\epsilon > 0$, there is N such that if $n > N$, then $|f(x_n) - f(x_0)| < \epsilon$, so $f(x_n) \to f(x_0)$. ✓

Intuitively: If x_n converges to x_0, then eventually x_n is in the red zone where $|x - x_0| < \delta$, and therefore $f(x_n)$ is $\epsilon-$close to $f(x_0)$, which forces $f(x_n)$ to converge to $f(x_0)$

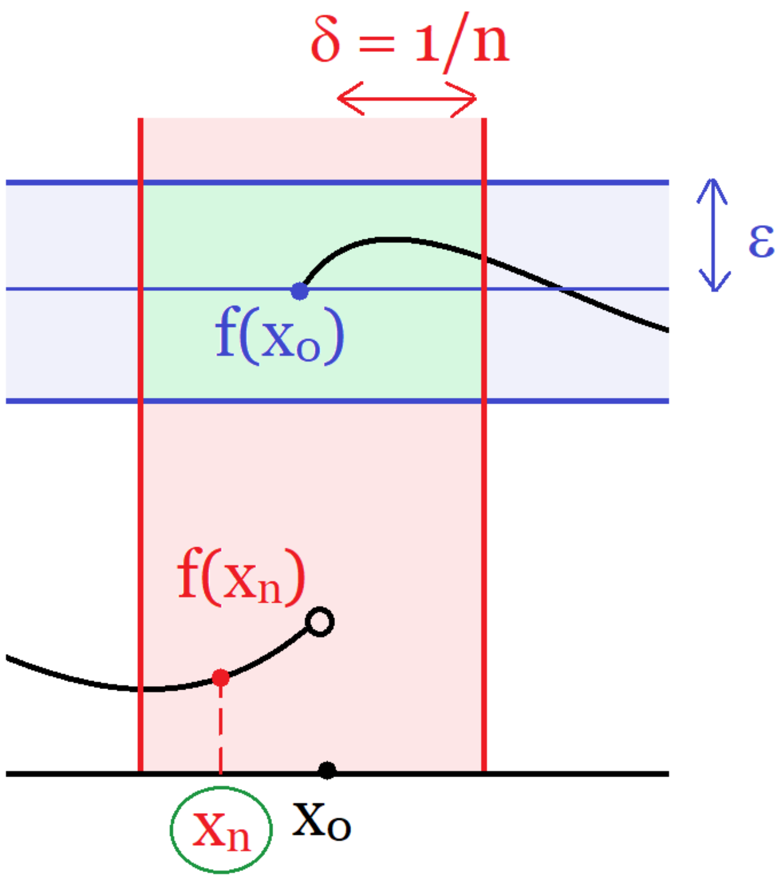


Proof: (Definition 1 ⇒ Definition 2)

We will show **(Not 2 ⇒ Not 1)**

Suppose $\epsilon - \delta$ definition fails, that is there is $\epsilon > 0$ such that for all $\delta > 0$, there is x such that $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \epsilon$.

The idea is simply to use the above definition but with $\delta = \frac{1}{n}$



With ϵ as above, for every n, with $\delta = \frac{1}{n}$, there is some x_n such that $|x_n - x_0| < \frac{1}{n}$ but $|f(x_n) - f(x_0)| \geq \epsilon$.

Since $|x_n - x_0| < \frac{1}{n}$, we get $x_n \to x_0$ by the Squeeze Theorem

But since $|f(x_n) - f(x_0)| \geq \epsilon$ for all n, we cannot have $f(x_n) \to f(x_0)$

Hence we found a sequence $x_n \to x_0$ but $f(x_n) \nrightarrow f(x_0)$ ✓ □

2. $f+g$ IS CONTINUOUS

Video: $f+g$ is continuous

Now let's prove some basic properties of continuous functions, such as $f+g$ is continuous or fg is continuous.

Fact 1:

If f and g are continuous at x_0, then $f+g$ is continuous at x_0

Proof using Definition 1: Let x_n be a sequence converging to x_0. Then, since f is continuous at x_0, we get $f(x_n) \to f(x_0)$ and, since g is continuous at x_0, we have $g(x_n) \to g(x_0)$. But, by the sum law for limits of sequences (see section 9), we get:

$$(f+g)(x_n) = f(x_n) + g(x_n) \to f(x_0) + g(x_0) = (f+g)(x_0) \checkmark$$

Hence $f+g$ is continuous at x_0 $\square$

Note: Notice how the result about $f+g$ follows from the corresponding result for sequences! This will be pretty much true for all our proofs involving Definition 1.

Proof using Definition 2: (do not skip!)

Let $\epsilon > 0$ be given

Then, since f is continuous at x_0, there is $\delta_1 > 0$ such that if $|x - x_0| < \delta_1$, then $|f(x) - f(x_0)| < \frac{\epsilon}{2}$.

And, since g is continuous at x_0, there is $\delta_2 > 0$ such that if $|x - x_0| < \delta_2$, then $|g(x) - g(x_0)| < \frac{\epsilon}{2}$.

But then, if $\delta = \min\{\delta_1, \delta_2\} > 0$, we get:

$$\begin{aligned} |(f+g)(x) - (f+g)(x_0)| &= |f(x) + g(x) - (f(x_0) + g(x_0))| \\ &= |f(x) - f(x_0) + g(x) - g(x_0)| \\ &\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \checkmark \end{aligned}$$

Hence $f + g$ is continuous at x_0 □

3. kf IS CONTINUOUS

As a tribute to KFC, let's prove that:

Fact 2:

If f is continuous at x_0, and k is a real number, then kf is continuous at x_0

Proof using Definition 1: If (x_n) is a sequence that converges to x_0, then, since f is continuous at x_0, $f(x_n) \to f(x_0)$, and therefore

$$(kf)(x_n) = k(f(x_n)) \to k(f(x_0)) = (kf)(x_0) \checkmark$$

And therefore kf is continuous at x_0 □

Proof using Definition 2: First of all, we may assume $k \neq 0$, because otherwise $kf = 0$, which is continuous.

Let $\epsilon > 0$, then, since f is continuous at x_0, there is $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \frac{\epsilon}{|k|}$ (we use absolute values because

k might be negative)

Then, with the same δ, if $|x - x_0| < \delta$, we get:

$$|(kf)(x) - (kf)(x_0)| = |kf(x) - kf(x_0)| = |k|\,|f(x) - f(x_0)| < |k| \left(\frac{\epsilon}{|k|}\right) = \epsilon \checkmark$$

Therefore kf is continuous at x_0 □

Aside: If you've taken linear algebra, notice that Fact 1 says that continuous functions are closed under addition, and Fact 2 says that they are closed under scalar multiplication. Therefore, the set of continuous functions forms a vector space!

Corollary:

If f and g are continuous at x_0, then $f - g$ is continuous at x_0

Proof: Since g is continuous at x_0, using Fact 2 above with $k = -1$, we get $-g = (-1)g$ is continuous at x_0.

Therefore, since f and $-g$ are continuous at x_0, by Fact 1, $f - g = f + (-g)$ is continuous at x_0 □

4. $|f|$ IS CONTINUOUS

In this small interlude, let's prove the following quick fact:

Fact 3:

If f is continuous at x_0, then $|f|$ is continuous at x_0

Proof using Definition 1: Suppose $x_n \to x_0$, then, since f is continuous at x_0, $f(x_n) \to f(x_0)$, and therefore $|f(x_n)| \to |f(x_0)|$ ✓

Hence $|f|$ is continuous at x_0 □

Proof using Definition 2: Let $\epsilon > 0$ be given. Then, since f is continuous at x_0, there is $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

With that same δ, if $|x - x_0| < \delta$, then by the reverse triangle inequality, which says $|a - b| \geq ||a| - |b||$, we have:

$$||f(x)| - |f(x_0)|| \leq |f(x) - f(x_0)| < \epsilon \checkmark$$

Therefore $|f|$ is continuous at x_0 □

5. fg IS CONTINUOUS

Video: fg is continuous

Now let's prove that the product of continuous functions is continuous:

Fact 4:

If f and g are continuous at x_0, then fg is continuous at x_0

Proof using Definition 1: Suppose $x_n \to x_0$. Then, since f is continuous at x_0, we have $f(x_n) \to f(x_0)$, and, since g is continuous at x_0, we have $g(x_n) \to g(x_0)$, and therefore, by the product law for limits (section 9), we have

$$(fg)(x_n) = (f(x_n))\,(g(x_n)) \to (f(x_0))\,(g(x_0)) = (fg)(x_0) \checkmark$$

Therefore fg is continuous at x_0 $\square$

Proof using Definition 2:

STEP 1: Scratchwork

We need to estimate:

$$\begin{aligned}|f(x)g(x) - f(x_0)g(x_0)| &= |f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)| \\ &= |f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0))| \\ &\leq |f(x)|\,|g(x) - g(x_0)| + |g(x_0)|\,|f(x) - f(x_0)|\end{aligned}$$

The $|f(x) - f(x_0)|$ and $|g(x) - g(x_0)|$ terms are good, since f and g are continuous at x_0. Moreover, the $|g(x_0)|$ term is good since it is constant.

The only bad term is $|f(x)|$ since it depends on x. For this, use the fact that, since f is continuous, $f(x)$ is close to $f(x_0)$ (which is constant)

Since f is continuous with $\epsilon = 1$, we get that there is $\delta_1 > 0$ such that if $|x - x_0| < \delta_1$, then $|f(x) - f(x_0)| < 1$, but then

$$|f(x)| = |f(x) - f(x_0) + f(x_0)| \leq |f(x) - f(x_0)| + |f(x_0)| < 1 + |f(x_0)|$$

Therefore, going back to our original inequality, we get:

$$\begin{aligned}|f(x)g(x) - f(x_0)g(x_0)| &\leq |f(x)|\,|g(x) - g(x_0)| + |g(x_0)|\,|f(x) - f(x_0)| \\ &\leq (|f(x_0)| + 1)\,|g(x) - g(x_0)| + |g(x_0)|\,|f(x) - f(x_0)|\end{aligned}$$

We are finally ready for our actual proof:

STEP 2: Actual Proof:

Let $\epsilon > 0$ be given

Then, since f is continuous at x_0, there is $\delta_1 > 0$ such that if $|x - x_0| < \delta_1$, then $|f(x) - f(x_0)| < 1$, and therefore $|f(x)| \leq |f(x_0)| + 1$ (as before)

Now since g is continuous at x_0, there is $\delta_2 > 0$ such that if $|x - x_0| < \delta_2$, then $|g(x) - g(x_0)| < \frac{\epsilon}{2(|f(x_0)|+1)}$

(the factor 2 is there because we have 2 terms)

Finally, since f is continuous at x_0, there is $\delta_3 > 0$ such that if $|x - x_0| < \delta_3$, then $|f(x) - f(x_0)| < \frac{\epsilon}{2(|g(x_0)|+1)}$

(we can't just divide by $|g(x_0)|$ since $g(x_0)$ might be 0)

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$, then if $|x - x_0| < \delta$, then we get:

$$\begin{aligned}
|(fg)(x) - (fg)(x_0)| &= |f(x)g(x) - f(x_0)g(x_0)| \\
&\leq (|f(x_0)| + 1)\,|g(x) - g(x_0)| + |g(x_0)|\,|f(x) - f(x_0)| \\
&< \cancel{(|f(x_0)| + 1)} \left(\frac{\epsilon}{2\cancel{(|f(x_0)| + 1)}} \right) + |g(x_0)| \left(\frac{\epsilon}{2\,(|g(x_0)| + 1)} \right) \\
&= \frac{\epsilon}{2} + \underbrace{\left(\frac{|g(x_0)|}{|g(x_0)| + 1} \right)}_{<1} \left(\frac{\epsilon}{2} \right) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon \checkmark
\end{aligned}$$

Therefore fg is continuous at x_0 $\square$

6. $\frac{f}{g}$ IS CONTINUOUS

Video: $\frac{f}{g}$ is continuous

In this section, we prove that quotients $\frac{f}{g}$ of continuous functions are continuous. For this, we need to first show that reciprocals $\frac{1}{f}$ of continuous functions are continuous.

Fact 5:

If $f \neq 0$ and f is continuous at x_0, then $\frac{1}{f}$ is continuous at x_0

Proof using Definition 1: If x_n is a sequence converging to x_0, then, since f is continuous at x_0, $f(x_n) \to f(x)$. By assumption $f(x_n) \neq 0$ for all n and $f(x) \neq 0$, so, by the results in section 9, $\frac{1}{f(x_n)} \to \frac{1}{f(x_0)}$ ✓

Therefore $\frac{1}{f}$ is continuous at x_0. □

Proof using Definition 2:

STEP 1: Scratchwork

This time we need to estimate

$$\left|\frac{1}{f(x)} - \frac{1}{f(x_0)}\right| = \left|\frac{f(x_0) - f(x)}{f(x)f(x_0)}\right| = \frac{|f(x) - f(x_0)|}{|f(x)|\,|f(x_0)|}$$

The $|f(x) - f(x_0)|$ term is good, and the $|f(x_0)|$ term is good as well

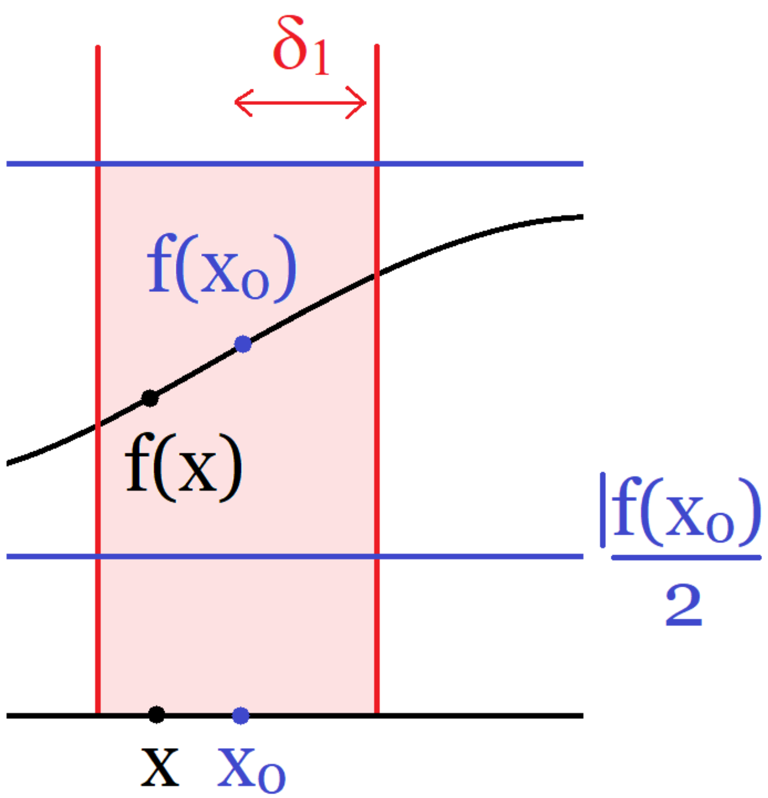
The only term we need to control is the $|f(x)|$ term.

Note: Since we want $\frac{1}{|f(x)|} <$ something, we need $|f(x)| >$ something!

Here we can't use the trick with $|f(x) - f(x_0)| < 1$, because here it depends on where x_0 is located (this will be clearer below)

That's why we need a more subtle estimate:

Since f is continuous at x_0, with $\epsilon = \frac{|f(x_0)|}{2} > 0$, there is δ_1 such that if $|x - x_0| < \delta_1$, then $|f(x) - f(x_0)| < \frac{|f(x_0)|}{2}$



(In the picture above, notice that in the red region, $f(x)$ is above $\frac{|f(x_0)|}{2}$)

But then, since we need $|f(x)| \geq$ something, using the *reverse* triangle inequality, we get

$$||f(x)| - |f(x_0)|| \leq |f(x) - f(x_0)| < \frac{|f(x_0)|}{2}$$

Therefore

$$-\frac{|f(x_0)|}{2} < |f(x)| - |f(x_0)| < -\frac{|f(x_0)|}{2}$$

And therefore

$$|f(x)| > |f(x_0)| - \frac{|f(x_0)|}{2} = \frac{|f(x_0)|}{2} > 0$$

(**THIS** step would have failed if we chose 1 instead of $\frac{|f(x_0)|}{2}$, we wouldn't get something positive)

$$\text{Hence } \frac{1}{|f(x)|} < \frac{2}{|f(x_0)|} \text{ (GOOD)}$$

Hence, going back to our original identity, we get

$$\begin{aligned} \left| \frac{1}{f(x)} - \frac{1}{f(x_0)} \right| = \frac{|f(x) - f(x_0)|}{|f(x)| \, |f(x_0)|} \leq \frac{|f(x) - f(x_0)|}{|f(x_0)|} \left(\frac{2}{|f(x_0)|} \right) \\ = |f(x) - f(x_0)| \left(\frac{2}{|f(x_0)|^2} \right) \overset{?}{<} \epsilon \end{aligned}$$

Which gives $|f(x) - f(x_0)| < \frac{\epsilon}{2} |f(x_0)|^2$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given

Then, since f is continuous at x_0, there is $\delta_1 > 0$ such that if $|x - x_0| < \delta_1$, then $|f(x) - f(x_0)| < \frac{|f(x_0)|}{2}$, which implies $|f(x)| > \frac{|f(x_0)|}{2}$, and therefore $\frac{1}{|f(x)|} < \frac{2}{|f(x_0)|}$

Moreover, since f is continuous at x_0, there is $\delta_2 > 0$ such that if $|x - x_0| < \delta_2$, then $|f(x) - f(x_0)| < \frac{\epsilon}{2} |f(x_0)|^2$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$, then, if $|x - x_0| < \delta$, then

$$\begin{aligned}
\left|\frac{1}{f(x)} - \frac{1}{f(x_0)}\right| &= \frac{|f(x) - f(x_0)|}{|f(x)|\,|f(x_0)|} \\
&\leq \left(\frac{|f(x) - f(x_0)|}{|f(x_0)|}\right)\left(\frac{2}{|f(x_0)|}\right) \\
&= |f(x) - f(x_0)| \left(\frac{2}{|f(x_0)|^2}\right) \\
&< \left(\frac{\epsilon \cancel{|f(x_0)|^2}}{\cancel{2}}\right)\cancel{\left(\frac{2}{|f(x_0)|^2}\right)} \\
&= \epsilon \checkmark
\end{aligned}$$

Hence $\frac{1}{f}$ is continuous at x_0 □

Corollary:

If f and g are continuous at x_0 with $g \neq 0$, then then $\frac{f}{g}$ is continuous at x_0

Proof: Since g is continuous at x_0 and $g \neq 0$, by the above, $\frac{1}{g}$ is continuous at x_0, and therefore, by the product law (Fact 4), $\frac{f}{g} = f\left(\frac{1}{g}\right)$ is continuous at x_0 □

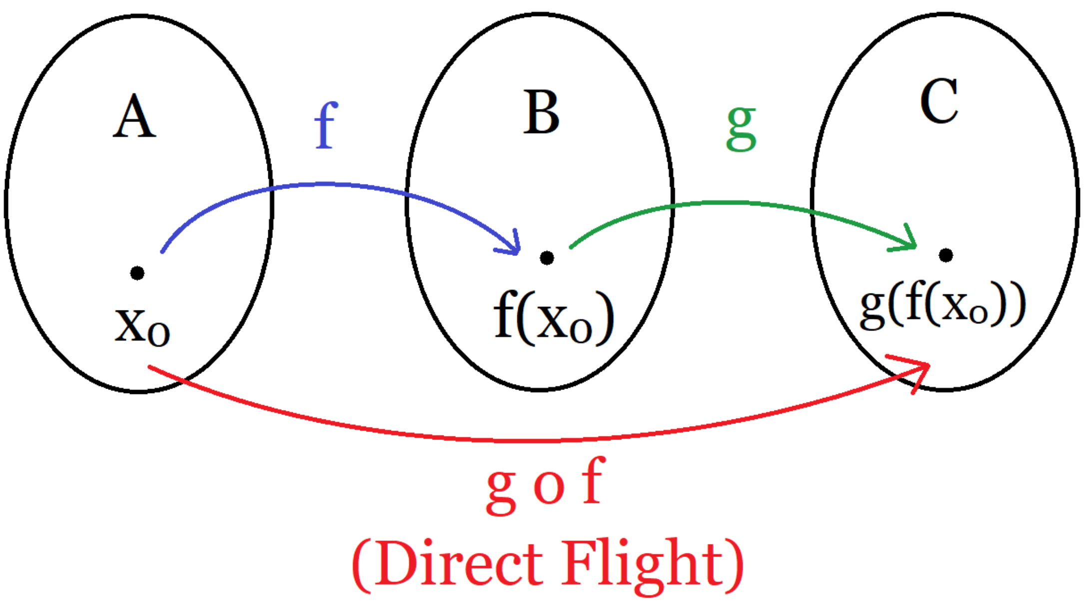
7. Chen Lu is continuous

Video: $g \circ f$ is continuous

Definition:

If A, B, C are subsets of $\mathbb{R}$ and $f : A \to B$ and $g : B \to C$ are functions, then the **composition** $g \circ f : A \to C$ is defined by

$$(g \circ f)(x) = g(f(x))$$



Analogy: If you think of f as a layover from A to B and g as a layover from B to C, then $g \circ f$ is a direct flight from A to C

Fact 6:

If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0

Proof using Definition 1: Suppose (x_n) is a sequence that converges to x_0. Then, since f is continuous at x_0, we have $f(x_n) \to f(x_0)$, but now, since g is continuous at $f(x_0)$, we have $g(f(x_n)) \to g(f(x_0))$, that is $(g \circ f)(x_n) \to (g \circ f)(x_0)$ ✓

And therefore $g \circ f$ is continuous at x_0 $\square$

Proof using Definition 2: Let $\epsilon > 0$ be given.

Since g is continuous at $f(x_0)$, there is $\delta' > 0$ such that

$$|x - f(x_0)| < \delta' \Rightarrow |g(x) - g(f(x_0))| < \epsilon$$

Since "$f(x)$" is more specific than "x", this implies that for all x,

$$|f(x) - f(x_0)| < \delta' \Rightarrow |g(f(x)) - g(f(x_0))| < \epsilon$$

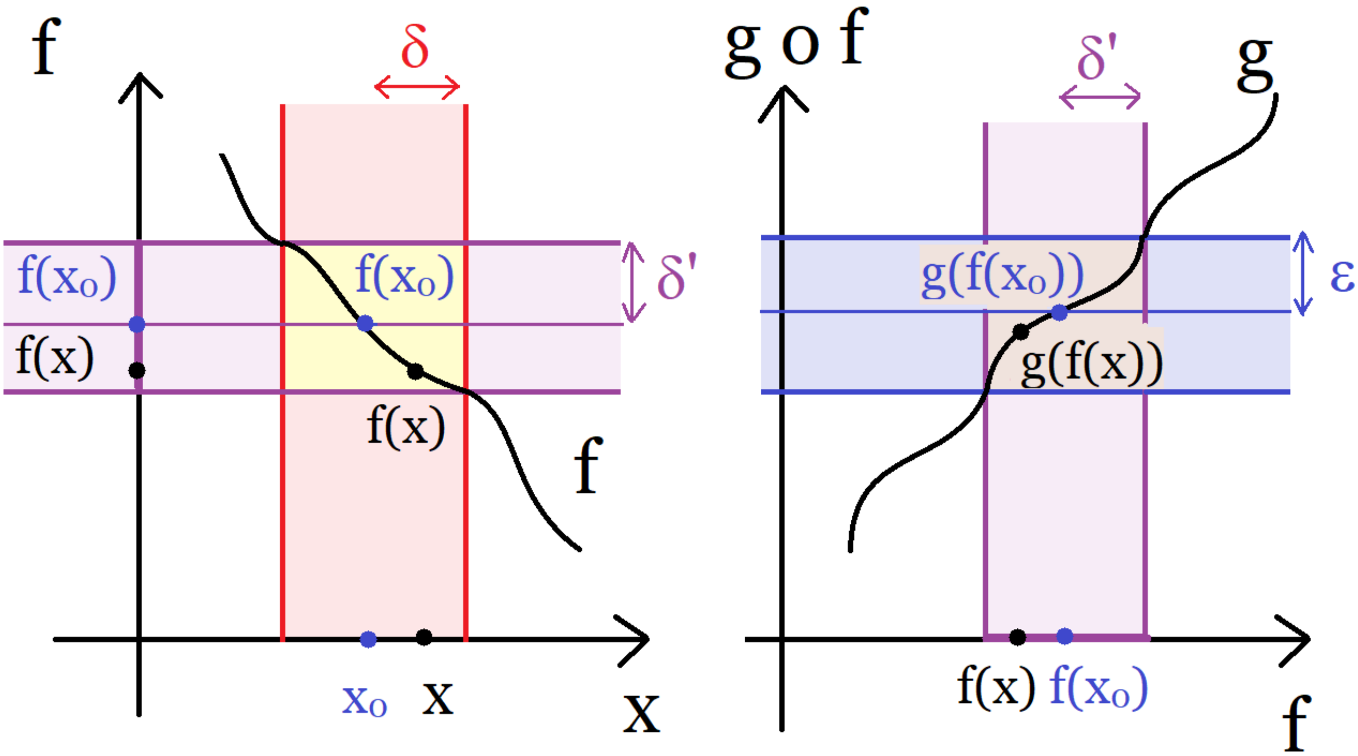
Since f is continuous at x_0, by $\epsilon - \delta$ with δ' instead of ϵ, there is $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \delta'$

So with δ as above, for all x, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \delta'$ and therefore

$$|(g \circ f)(x) - (g \circ f)(x_0)| = |g(f(x)) - g(f(x_0))| < \epsilon \checkmark$$

Therefore $g \circ f$ is continuous at x_0 $\square$

Intuitively: We need $g(f(x))$ to be in the good region (in blue on the right), this can be achieved by making $f(x)$ close to $f(x_0)$ (purple region on the right) since g is continuous, and this, in turn, can be achieved by making x close to x_0 (red region on the left) since f is continuous.



8. $\max(f, g)$ IS CONTINUOUS

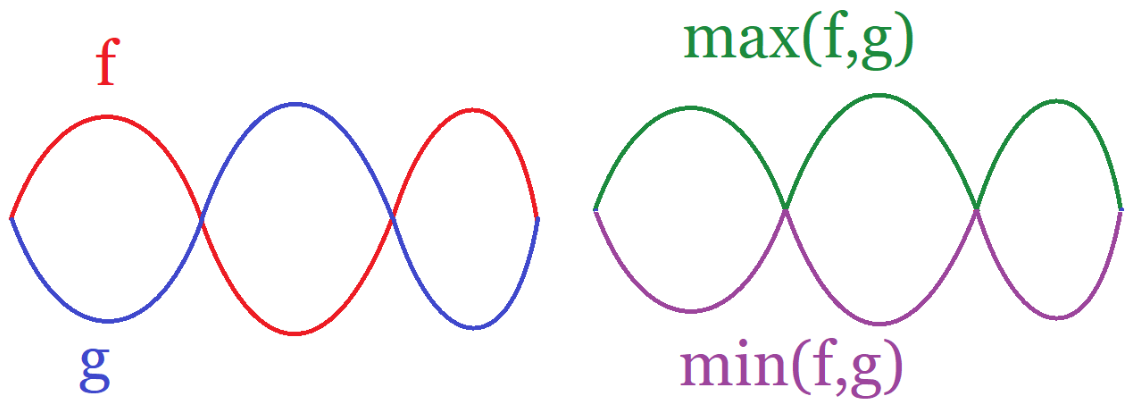
Video: max is continuous

Finally, let's show that the maximum of f and g is continuous.

Definition:

$$\max(f,g)(x) = \begin{cases} f(x) \text{ if } f(x) \geq g(x) \\ g(x) \text{ if } g(x) \geq f(x) \end{cases}$$

In other words, at each x, $\max(f,g)$ is just the bigger one of $f(x)$ and $g(x)$



Fact 7:

If f and g are continuous at x_0, then $\max(f,g)$ is continuous at x_0

The proof of this relies on the following explicit formula for $\max(f,g)$

Claim:

$$\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$$

Proof of Claim:

Case 1: $f(x) \geq g(x)$

Then $\max(f,g) = f(x)$, but also, since $f(x) - g(x) \geq 0$, we have $|f(x) - g(x)| = f(x) - g(x)$, and so

$$\begin{aligned}
\frac{1}{2}(f(x)+g(x)) + \frac{1}{2}|f(x)-g(x)| =& \frac{1}{2}(f(x)+g(x)) + \frac{1}{2}(f(x)-g(x)) \\
=& \frac{1}{2}(f(x)+g(x)+f(x)-g(x)) \\
=& \frac{1}{2}(2f(x)) \\
=& f(x) \checkmark
\end{aligned}$$

Case 2: $g(x) \leq f(x)$

Similar, except you use $|f(x) - g(x)| = g(x) - f(x)$ since $f(x) - g(x) \leq 0$ here ✓ □

Proof of Fact: Since f and g are continuous at x_0, $f + g$ is continuous at x_0, and therefore $\frac{1}{2}(f+g)$ is continuous at x_0.

But also $f - g$ is continuous at x_0, and therefore $|f - g|$ is continuous at x_0, and hence $\frac{1}{2}|f-g|$ is continuous at x_0, and therefore:

$$\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$$

is continuous at x_0 (as the sum of two continuous functions) □

Remark: Similarly, you can define

Definition:

$$\min(f,g)(x) = \begin{cases} f(x) \text{ if } f(x) \leq g(x) \\ g(x) \text{ if } g(x) \leq f(x) \end{cases}$$

And similarly you can show

Fact:

If f and g are continuous at x_0, then $\min(f,g)$ is continuous at x_0

Proof: See Homework for details, but you either show (similar to above) that

$$\min(f,g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$$

Or use that

$$\min(f,g) = -\max(-f,-g) \quad \square$$

(Compare this to $\inf(S) = -\sup(-S)$ from Chapter 1)