LECTURE 16: MAX AND MIN VALUES (I)

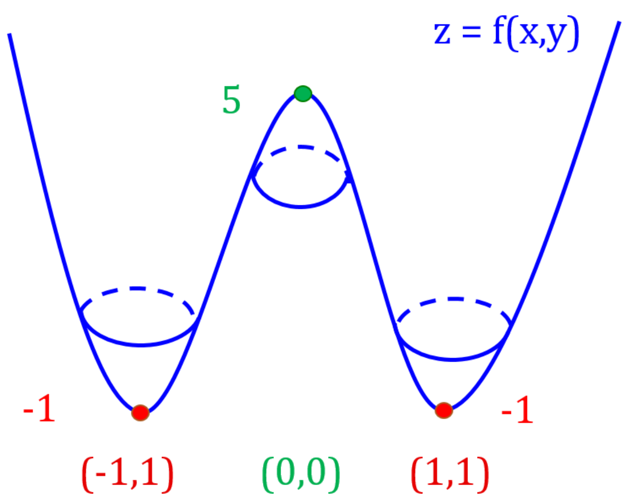
Welcome to the highlight of our partial derivatives chapter! Today we'll use partial derivatives to figure out when a function has a local maximum or minimum.

1. Maximum and Minimum Values

Goal: Find the local maxima and minima of a function

Example 1:

In the picture below, f has a local maximum of 5 at $(0,0)$ and a local minimum of -1 at $(-1,1)$ and at $(1,1)$



Date: Monday, October 4, 2021.

The good news is that it's similar to single-variable calculus, but this time with a little twist

2. Critical Points

Recall:

In single-variable calculus, you first had to find the critical points of f, where $f'(c) = 0$.

Here it's the same thing:

Example 2:

Find the critical points of $f(x, y) = x^4 + y^4 - 4x - 32y + 14$

Definition:

(a, b) is a critical point of $f_x(a, b) = 0$ **and** $f_y(a, b) = 0$

(In other words $\nabla f = \langle 0, 0 \rangle$)

$$\begin{aligned} f_x =& 4x^3 - 4 = 0 \Rightarrow 4x^3 = 4 \Rightarrow x^3 = 1 \Rightarrow x = 1 \\ f_y =& 4y^3 - 32 = 0 \Rightarrow 4y^3 = 32 \Rightarrow y^3 = 8 \Rightarrow y = 2 \end{aligned}$$

Therefore, the only critical point is $(1, 2)$.

3. Second Derivatives

Recall:

Then you had to use the second derivative test: for every critical point c, calculate $f''(c)$ and find its sign

The only problem is that $f(x,y)$ has 4 second derivatives: $f_{xx}, f_{xy}, f_{yx}, f_{yy}$, so we need to put them together in one quantity.

Example 3:

Find $D(1,2)$, where $f(x,y) = x^4 + y^4 - 4x - 32y + 14$.

(Here $(1,2)$ is the critical point that we found)

Definition:

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

(D as in **D**eterminant, put all the second derivatives in a table and take the determinant)

Found $f_x = 4x^3 - 4$ and $f_y = 4y^3 - 32$

$$\begin{array}{cc} f_{xx} = 12x^2 & f_{xy} = 0 \\ f_{yx} = 0 & f_{yy} = 12y^2 \end{array}$$

$$D(x,y) = \begin{vmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{vmatrix}$$

⚠ **FIRST** plug in $(1,2)$ and **THEN** calculate the determinant! Otherwise this will be a hot mess!

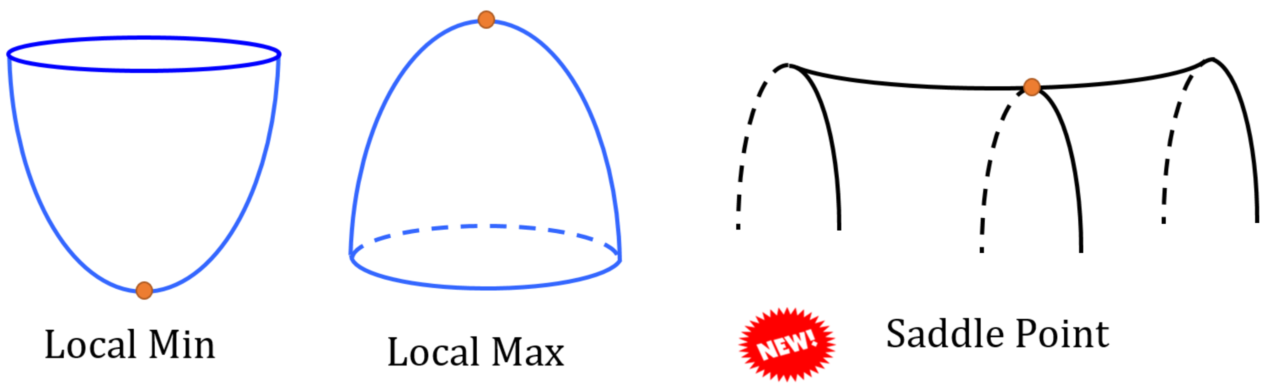
$$D(1,2) = \begin{vmatrix} 12(1)^2 & 0 \\ 0 & 12(2)^2 \end{vmatrix} = \begin{vmatrix} 12 & 0 \\ 0 & 48 \end{vmatrix} = (12)(48) - (0)(0) = 576 > 0$$

(Here we used the fact that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$)

Now let's see what $D(1,2) > 0$ tells us about local max/min.

4. The Saddle Point Test

Note: In 2 dimensions, a function can have: a local min, a local max, or a saddle point (goes up and down at the same time)



Turns out there is an easy way to figure out if a function has a saddle point or not:

Saddle Point Test:

If $D(1,2) < 0$, then f has a saddle at $(1,2)$

Mnemonic: Think of "negative" as "sad," just like a **sad**dle!

STEP 3: Here: $D(1,2) = 576 > 0$, so **NOT** a saddle point

5. The Second Derivative Test

What if you have a critical point and it's not a saddle, like $(1,2)$ above? Now we're finally ready for the second derivative test

Recall:

In single-variable calculus, In single variable-calculus, $f''(c) < 0 \Rightarrow$ local max and $f''(c) > 0 \Rightarrow$ local min

Second Derivative Test

Assume $D(1,2) > 0$ (not a saddle)

(1) If $f_{xx}(1,2) > 0$, then f has a local min at $(1,2)$

(2) If $f_{xx}(1,2) < 0$, then f has a local max at $(1,2)$

STEP 4: Found

$$D(1,2) = \begin{vmatrix} 12 & 0 \\ 0 & 48 \end{vmatrix} = 576 > 0$$

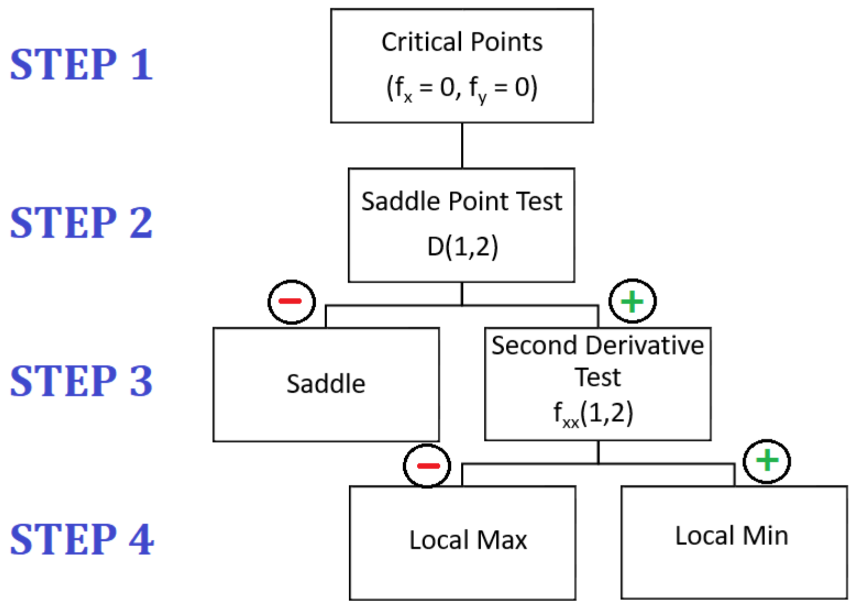
Also $f_{xx}(1,2) = 12 > 0$, so f has a **local min** at $(1,2)$.

Answer: The local min of f is:

$$f(1,2) = 1^4 + 2^4 - 4(1) - 32(2) + 14 = -37$$

6. Putting it together

Suggestion: First apply the saddle test, and **then** apply the second derivative test



Note: If you find that this process is weird and convoluted, it's because it is! There is a very elegant (and more correct) way of doing this, but it requires eigenvalues from linear algebra. If you're interested in the *true* second derivative test, check out this video

Video: The True Second Derivative Test (optional)

Example 4: (Good quiz/exam question)

Find all the local max, min, and saddle points of

$$f(x,y) = x^3 - 3x + 3xy^2$$

STEP 1: Critical Points

$$f_x = 3x^2 - 3 + 3y^2 = 0$$
$$f_y = 6xy = 0 \Rightarrow xy = 0 \Rightarrow x = 0 \text{ or } y = 0$$

Case 1: $x = 0$, then:

$$f_x = 3(0)^2 - 3 + 3y^2 = -3 + 3y^2 = 0 \Rightarrow 3y^2 = 3 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$$

This gives us the critical points $(0, \pm 1)$

Case 2: $y = 0$, then

$$f_x = 3x^2 - 3 + 3(0)^2 = 3x^2 - 3 = 0 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

This gives us the critical points $(\pm 1, 0)$

Hence there are 4 critical points: $(\pm 1, 0), (0, \pm 1)$, and we need to apply the second derivative tests on all of them.

STEP 2: $D(x, y)$

Recall $f_x = 3x^2 - 3 + 3y^2$ and $f_y = 6xy$

$f_{xx} = 6x,\ f_{xy} = 6y,\ f_{yx} = 6y,\ f_{yy} = 6x$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 6y \\ 6y & 6x \end{vmatrix}$$

STEP 3: Saddle Point/Second Derivative Tests

Point 1: $(0, 1)$

$$D(0,1) = \begin{vmatrix} 0 & 6 \\ 6 & 0 \end{vmatrix} = -36 < 0 \Rightarrow \text{ Saddle Point}$$

Point 2: $(0,-1)$

$$D(0,-1) = \begin{vmatrix} 0 & -6 \\ -6 & 0 \end{vmatrix} = -(-6)(-6) = -36 < 0 \Rightarrow \text{ Saddle Point}$$

Point 3: $(1,0)$

$$D(1,0) = \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36 > 0 \text{ and } f_{xx}(1,0) = 6 > 0 \Rightarrow \text{ Local Min}$$

Point 4: $(-1,0)$

$$D(-1,0) = \begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix} = 36 > 0 \text{ and } f_{xx}(-1,0) = -6 < 0 \Rightarrow \text{ Local Max}$$

STEP 4: Conclusion

Saddle Points:

$$\begin{aligned} f(0,1) =& 0^3 - 3(0) + 3(0)(1)^2 = 0 \\ f(0,-1) =& 0^3 - 3(0) + 3(0)(-1)^2 = 0 \end{aligned}$$

Local Min:

$$f(1,0) = 1^3 - 3(1) + 3(1)(0)^2 = -2$$

Local Max:

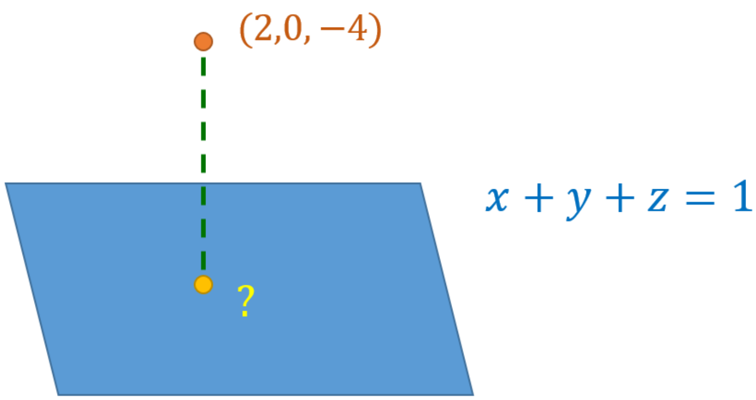
$$f(-1,0) = (-1)^3 - 3(-1) - 3(-1)(0)^2 = -1 + 3 = 2$$

7. Distance Problem

Just like in single-variable calculus, there are lots of word problems involving max and min. This one and the next one are two very classical problems.

Example 5:

Find the point on the plane $x + y + z = 1$ that is closest to $(2, 0, -4)$



⚠ We can't just use the formula for the distance between a point and a plane (from the previous chapter), because that formula just gives you the distance, not the point that is closest.

STEP 1: Find $f(x, y)$

Recall: The distance between (x, y, z) and $(2, 0, -4)$ is

$$d = \sqrt{(x-2)^2 + y^2 + (z+4)^2}$$

We want to write this in terms of x and y only

By the equation of the plane, we have $x + y + z = 1 \Rightarrow z = 1 - x - y$

Therefore:

$$d = \sqrt{(x-2)^2 + y^2 + (1 - x - y + 4)^2}$$

We *could* do the problem using $f(x, y) = \sqrt{\cdots}$ but notice that if d has a minimum at a point, then so does d^2 ! Therefore, we let

$$f(x, y) = d^2 = (x-2)^2 + y^2 + (5 - x - y)^2$$

And our goal is to minimize f

STEP 2: Critical Points

$$\begin{aligned} f_x =& 2(x-2) + 2(5-x-y)(-1) \\ =& 2x - 4 - 10 + 2x + 2y \\ =& 4x + 2y - 14 = 0 \end{aligned}$$

$$\begin{aligned} f_y =& 2y + 2(5-x-y)(-1) \\ =& 2y - 10 + 2x + 2y \\ =& 2x + 4y - 10 = 0 \end{aligned}$$

Therefore we need to solve the system

$$\begin{cases} 4x + 2y = 14 \\ 2x + 4y = 10 \end{cases} \Rightarrow \begin{cases} 2x + y = 7 \\ x + 2y = 5 \end{cases}$$

The first equation gives us $y = 7 - 2x$ and plugging this into the second equation, we get

$$
\begin{aligned}
x + 2(7 - 2x) &= 5 \\
x + 14 - 4x &= 5 \\
-3x &= 5 - 14 \\
-3x &= -9 \\
x &= 3
\end{aligned}
$$

And hence $y = 7 - 2(3) = 7 - 6 = 1$, so $y = 1$

Therefore the only critical point is $(3, 1)$

STEP 3: $f_x = 4x + 2y - 14$ and $f_y = 2x + 4y - 10$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix}$$

$$D(3,1) = \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} = 12 > 0$$

And since $f_{xx} = 4 > 0$ we get that $f(x,y)$ has a local min at $(3, 1)$

STEP 4: Answer:

$$x = 3, y = 1, z = 1 - x - y = 1 - 3 - 1 = -3$$

Hence the point on the plane closest to $(2, 0, -4)$ is $(3, 1, -3)$

Note: The distance from $(2, 0, -4)$ to the plane $x + y + z = 1$ is then just the distance from $(2, 0, -4)$ to $(3, 1, -3)$, which is:

$$\sqrt{(3-2)^2 + (1-0)^2 + (-3+4)^2} = \sqrt{1+1+1} = \sqrt{3}$$

This is basically another derivation of getting the distance between a point and a plane.