LECTURE 16: THE IMPLICIT FUNCTION THEOREM

On the other side of the same coin is the Implicit Function Theorem.

1. Motivation

Goal: Suppose you have an equation of the form $F(x,y) = 0$, can you solve for one variable in terms of the other one(s)?

Example 1: Let $F(x,y) = x^2 + y^2 - 1 = 0$, that is $x^2 + y^2 = 1$. Then you can solve for y in terms of x because $y = \pm\sqrt{1-x^2}$. This expression fails precisely when $y = 0$ that is when $F_y = 0$ (this is the derivative of F with respect to the variable you want to solve for)

Moreover, we can calculate $\frac{dy}{dx}$ in terms of partial derivatives:

$$\begin{aligned}
\left(x^2 + y^2 - 1\right)' &= (0)' \\
2x + 2y\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{2x}{2y} \\
\frac{dy}{dx} &= -\frac{F_x}{F_y}
\end{aligned}$$

Notice how the x and y get switched in the right-hand-side. Again, notice how this is defined when $F_y \neq 0$

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Example 2: To prep for the notation of the Implicit Function Thm:

Let $n = 3$ (number of x variables) and $m = 2$ (number of y variables), and define

$$F : \mathbb{R}^{3+2} \to \mathbb{R}^2 \text{ by } F = (F_1, F_2) \text{ where}$$

$$\begin{aligned} F_1(x_1, x_2, x_3, y_1, y_2) =& x_1 y_2 - 4x_2 + 3 + 2e^{y_1} \\ F_2(x_1, x_2, x_3, y_1, y_2) =& 2x_1 - x_3 + y_2 \cos(y_1) - 6y_1 \end{aligned}$$

Notice $F(x_0, y_0) = 0$ where $x_0 = (3, 2, 7)$ and $y_0 = (0, 1)$

Question: Can we solve for y in terms of x, for x near $x_0 = (3, 2, 7)$?

The implicit function theorem says yes provided that "$F_y \neq 0$" (the derivative with respect to the variable you want to solve for is nonzero)

$$[F'(x, y)] = \left[\begin{bmatrix} y_2 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} \mid \begin{bmatrix} 2e^{y_1} & x_1 \\ -y_2 \sin(y_1) - 6 & \cos(y_1) \end{bmatrix} \right] = \begin{bmatrix} F_x & \mid & F_y \end{bmatrix}$$

$$[F'(x_0, y_0)] = \left[\begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} \mid \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix} \right] = \begin{bmatrix} F_x(x_0, y_0) & \mid & F_y(x_0, y_0) \end{bmatrix}$$

Here all you need to check here is that $F_y(x_0, y_0)$ is invertible, but

$$|F_y(x_0, y_0)| = \begin{vmatrix} 2 & 3 \\ -6 & 1 \end{vmatrix} = 2 + 18 = 20 \neq 0 \text{ YES}$$

Then the implicit function theorem then that there is a function $y = G(x)$ from a neighborhood W of $x_0 = (3, 2, 7)$ (x variables) to $\mathbb{R}^m$ such that $F(x, G(x)) = 0$ (the equation is satisfied)

Moreover, we can calculate $G'(3,2,7)$ via

$$G'(3,2,7) = -\left(F_y\right)^{-1} F_x = -\begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} = -\frac{1}{20}\begin{bmatrix} 5 & 4 & -3 \\ -10 & 12 & 2 \end{bmatrix}$$

Looking at the $(1,2)$ entry for example, this tells us $\frac{\partial y_1}{\partial x_2} = -\frac{4}{20} = -\frac{1}{5}$

Compare this once again with the $\frac{dy}{dx} = -\frac{F_x}{F_y}$ condition from Example 1.

2. The Implicit Function Theorem

Implicit Function Theorem:

Suppose $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is C^1 and $F(x_0, y_0) = 0$ for some (x_0, y_0).

If $\det F_y(x_0, y_0) \neq 0$, then there is an open neighborhood U of (x_0, y_0) and an open neighborhood W of x_0 and a function $G : W \to \mathbb{R}^m$ differentiable at x_0 such that

$$\{(x,y) \in U \mid F(x,y) = 0\} = \{(x, G(x)) \mid x \in W\}$$

$$\text{Moreover } G'(x_0) = -\left(F_y(x_0,y_0)\right)^{-1} F_x(x_0,y_0)$$

In other words, if the derivative with respect to the variable you want to solve for is invertible, then the equation $F(x,y) = 0$ is locally the graph of a function $y = G(x)$.

Application: This theorem is extremely useful in PDEs. Lots of PDEs, especially first-order ones, are usually given by implicit equations of the form $F(x, u, \nabla u) = 0$. The implicit function theorem can then be used to solve for u in terms of x, provided some "nondegeneracy" condition holds, which is usually equivalent to the assumption above.

Proof:[1]

Surprisingly, the Implicit function theorem and Inverse function theorem are equivalent (notice they both solve for one variable in terms of another one), so our goal is to apply the Inverse Function Theorem to a cleverly designed function

STEP 1: Given our F, define $f : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ by

$$f(x, y) = (x, F(x, y))$$

Goal: Apply the Inverse function theorem to f at $a = (x_0, y_0)$

First show that $f'(a)$ is invertible. However

$$[f'(x, y)] = \begin{bmatrix} I_{n\times n} & 0_{n\times m} \\ F_x & F_y \end{bmatrix}$$

$$\text{Hence } [f'(a)] = \begin{bmatrix} I_{n\times n} & 0_{n\times m} \\ F_x(x_0, y_0) & F_y(x_0, y_0) \end{bmatrix}$$

It then follows from cofactor expansion along the first n rows that

$$\det[f'(a)] = \det[F_y(x_0, y_0)] \neq 0$$

Where the last step follows precisely because $F_y(x_0, y_0)$ is invertible

STEP 2: Hence $f'(a)$ is invertible and therefore by the Inverse Function Theorem there is an open set U containing (x_0, y_0) and an open set V containing $f(x_0, y_0)$ such that $f : U \to V$ is invertible.

[1] The proof is taken from this website

Moreover, $f^{-1} : V \to U$ is differentiable at $f(x_0, y_0)$ and

$$\left(f^{-1}\right)'(f(x_0, y_0)) = \left(f'(x_0, y_0)\right)^{-1}$$

Note: $f(x_0, y_0) = (x_0, F(x_0, y_0)) = (x_0, 0)$

Write f^{-1} in terms of components as $f^{-1} =: (h, g)$

STEP 3: Define W and G as follows:

$$W =: \{x \in \mathbb{R}^n \mid (x, 0) \in V\}$$

(Think of it kind of like an x-axis of V)

$$G(x) =: g(x, 0) \text{ for } x \in W$$

Notice W is nonempty since $x_0 \in W$ and W is open since W is just a projection of V on $\mathbb{R}^n$.

Since f^{-1} is differentiable at $(x_0, 0)$ and g is a component of f^{-1} it follows that g is differentiable at $(x_0, 0) \in V$ and so G is differentiable at x_0.

STEP 4: Let's show

$$\{(x, y) \in U \mid F(x, y) = 0\} = \{(x, G(x)) \mid x \in W\}$$

Let A be the left hand side and B be the right-hand-side, and show each set is contained in the other.

$A \subseteq B$: If $(x, y) \in A$ then $(x, y) \in U$ and $F(x, y) = 0$ from which it follows that $f(x, y) = (x, \underbrace{F(x, y)}_{0}) = (x, 0)$

Since $(x, 0) \in V$ (range of f), by definition $x \in W$ and from $f(x, y) = (x, 0)$ we get $(x, y) = f^{-1}(x, 0) = (h(x, 0), g(x, 0))$

Comparing components, this implies $y = g(x, 0) = G(x)$ which implies that $(x, y) = (x, G(x))$ and since we've shown $x \in W$, we get that $(x, y) \in B$

Hence $A \subseteq B$ and similarly we have $B \subseteq A$

STEP 5: The only thing left to show is the formula for the derivatives

Note: $G(x_0) = y_0$ because $f(x_0, y_0) = (x_0, 0)$ implies $f^{-1}(x_0, 0) = (x_0, y_0)$ and comparing the second component we get $g(x_0, 0) = y_0$ so $G(x_0) = y_0$

Since $F(x, G(x)) = 0$ for all $x \in W$, F is differentiable at $(x_0, G(x_0)) = (x_0, y_0)$, and G is differentiable at x_0, by the Chain Rule, we have

$$\begin{aligned} (F(x, G(x)))' &= 0 \\ F_x(x_0, y_0) + F_y(x_0, y_0) G'(x_0) &= 0 \\ F_y(x_0, y_0) G'(x_0) &= -F_x(x_0, y_0) \\ G'(x_0) &= -\left(F_y(x_0, y_0)\right)^{-1} F_x(x_0, y_0) \quad \square \end{aligned}$$

3. The Rank Theorem

In the inverse function theorem, we assumed $f'(a)$ is invertible, and we were able to find a local inverse of f.

Question: What if $f'(a)$ is not invertible?

Then under some assumptions, the Rank Theorem says that, although not invertible, f is locally like a projection.

Definition: f and g are C^1-equivalent if there are diffeomorphisms α and β such that $\beta \circ f = g \circ \alpha$ (see picture in lecture)

In other words, if you ignore α and β, we get $f = g$

Definition: If T is a linear transformation, then $\text{rank}\,(T) = \dim(Col(T))$.

The rank measures the true size of a linear transformation. For example, the 0 transformation has rank 0, but an invertible linear transformation on $\mathbb{R}^n$ has rank n

Rank Theorem: Suppose there is r such that $\text{rank}\,(f'(x)) = r$ for all x, then locally f is C^1 equivalent to a projection on $r-$dimensional space

Example 1: If f' has constant rank 1, then locally f looks like the projection $P(x, y, z) = (x, 0, 0)$ (see picture in lecture)

Example 2: If f' has constant rank 2, then locally f looks like the projection $P(x, y, z) = (x, y, 0)$ (see picture in lecture)

What makes this amazing is that it tells you about the dimension of the space just by looking at tangent planes. For example, the tangent plane of the sphere in $\mathbb{R}^3$ has rank 2 (spanned by 2 vectors), which explains why that sphere is $2-$dimensional.

As another example, the radial projection $P : \mathbb{R}^3 \to S^2$ with $P(v) = \frac{v}{|v|}$ has constant rank 2 and is locally indistinguishable from linear projection of $\mathbb{R}^3$ to the (x, y) plane.