LECTURE 17: PROPERTIES OF CONTINUITY

Today: We'll prove two of the three *Value Theorems* used in Calculus: The Extreme Value Theorem and the Intermediate Value Theorem. The Mean Value Theorem will be proven in Chapter 5.

Note: To give you a break, today we will not use any $\epsilon - \delta$ ☺

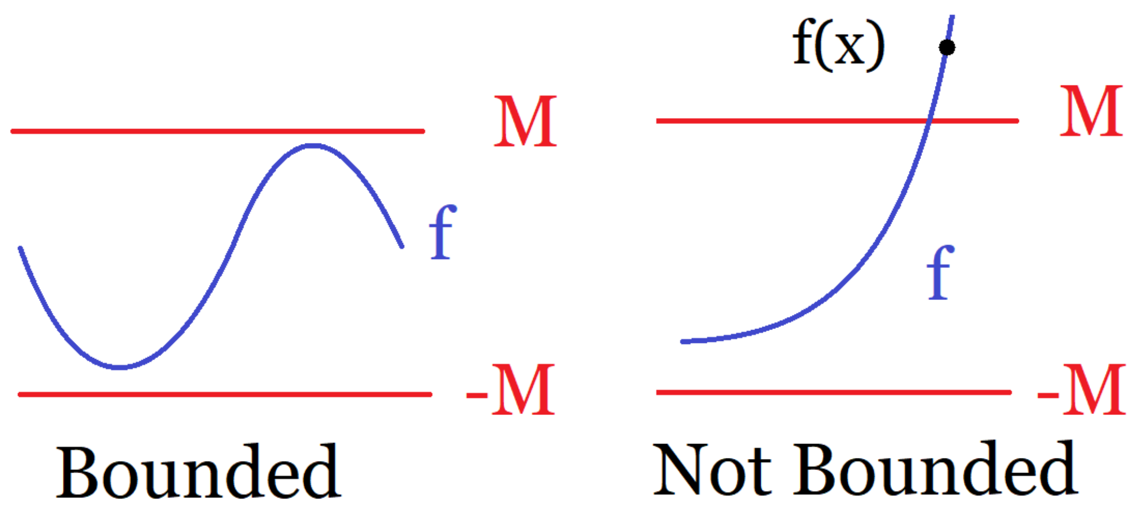
1. Bounded Functions

Video: Bounded Functions

As a warm-up, let's show that continuous functions are bounded

Defintion:

f is **bounded** if there is $M > 0$ such that for all x, $|f(x)| \leq M$



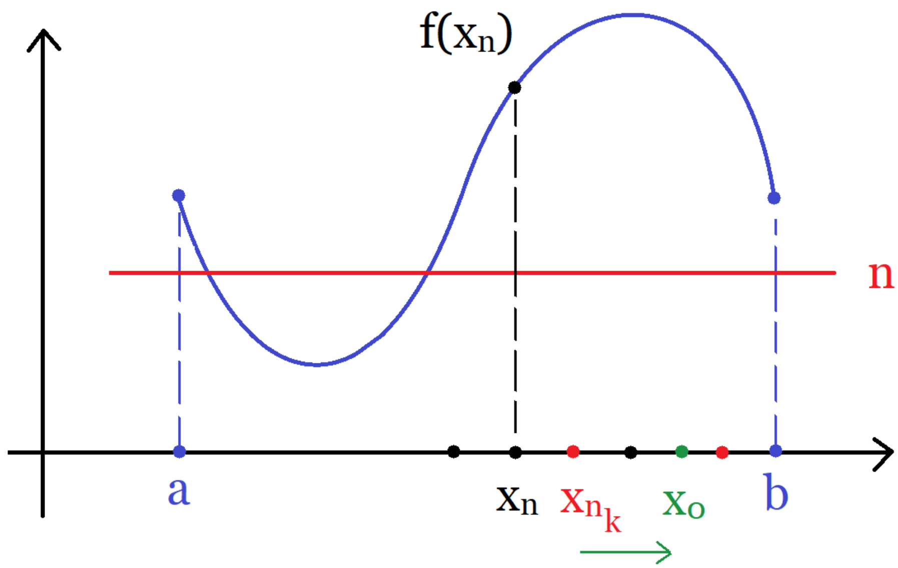
Date: Tuesday, October 26, 2021.

(This is similar to the definition of sequences being bounded)

Fact:

If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded

Proof: Suppose not. Then for all $n \in \mathbb{N}$, there is some $x_n \in [a, b]$ with $|f(x_n)| > n$ (this is just the negation with $M = n$)



Since $x_n \in [a, b]$, the sequence (x_n) is bounded.

Therefore, by Bolzano-Weierstraß, (x_n) has a convergent subsequence (x_{n_k}) that converges to some $x_0 \in [a, b]$.

Since $x_{n_k} \to x_0$ and f is continuous, we have $f(x_{n_k}) \to f(x_0)$ and so $|f(x_{n_k})| \to |f(x_0)|$

On the other hand, since $|f(x_n)| > n$ for all n, we have $|f(x_n)| \to \infty$. This is true for the subsequence $f(x_{n_k})$ as well, so $|f(x_{n_k})| \to \infty$

Comparing the two limits, we get $|f(x_0)| = \infty$, which is absurd $\Rightarrow\Leftarrow$

$\square$

2. The Extreme Value Theorem

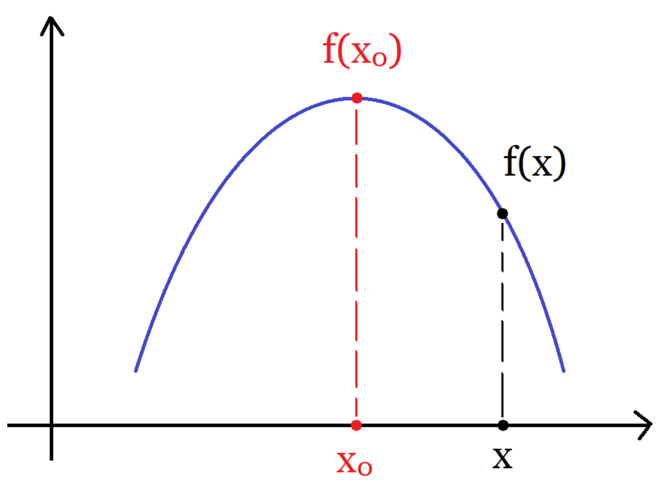
Video: The Extreme Value Theorem

The Extreme Value Theorem is one of the unsung heroes in Calculus. It says that any continuous function f on $[a, b]$ must have a max and a min. Without this, optimization problems would be impossible!

Defintion:

f has a **maximum** on $[a, b]$ if there is $x_0 \in [a, b]$ such that $f(x_0) \geq f(x)$ for all $x \in [a, b]$

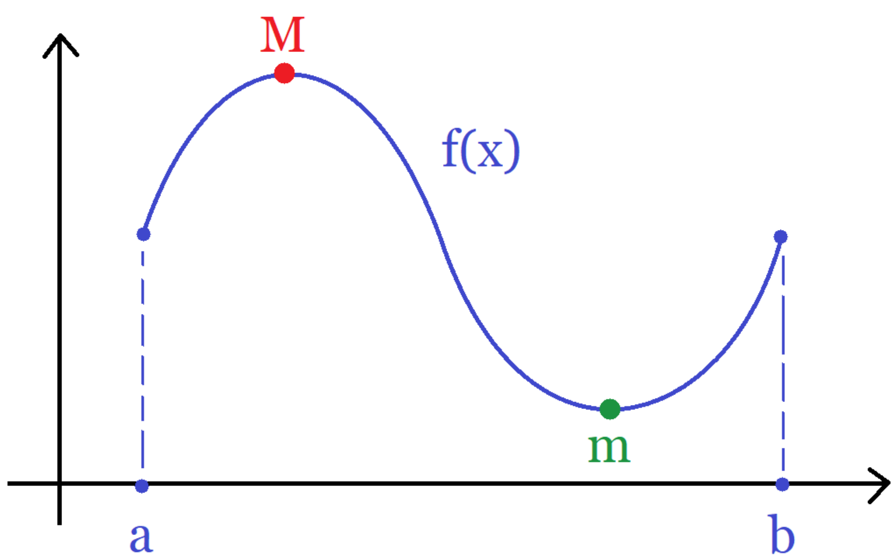
(Similarly for minimum)



Important: *By definition*, the maximum has to be attained. In other words, there must be some x_0 such that $f(x_0)$ is that maximum!

Extreme Value Theorem:

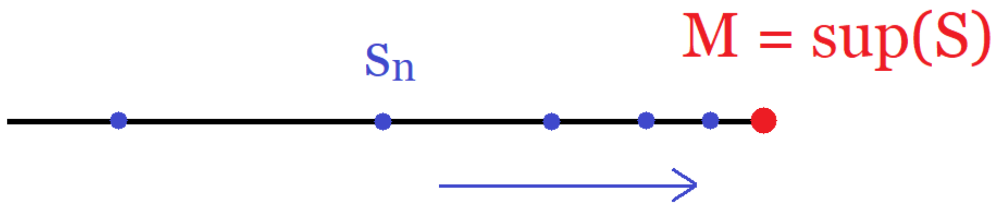
Suppose $f : [a, b] \to \mathbb{R}$ is continuous, then f has a maximum and a minimum on $[a, b]$



First, let's state a Useful Lemma that will be useful both here and for the Intermediate Value Theorem. The proof was on the homework.

Useful Lemma:

If S is bounded above, then there is a sequence (s_n) in S that converges to $\sup(S)$

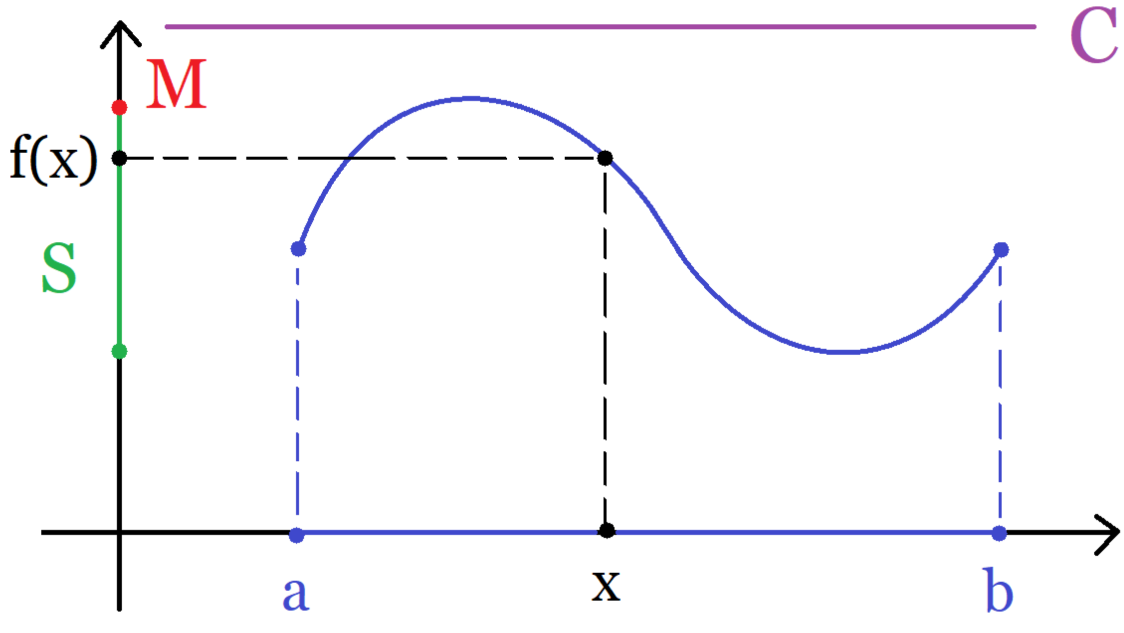


Proof of the Extreme Value Theorem:

Note: It's enough to show that f has a maximum, since we can repeat the same proof with $-f$ instead of f

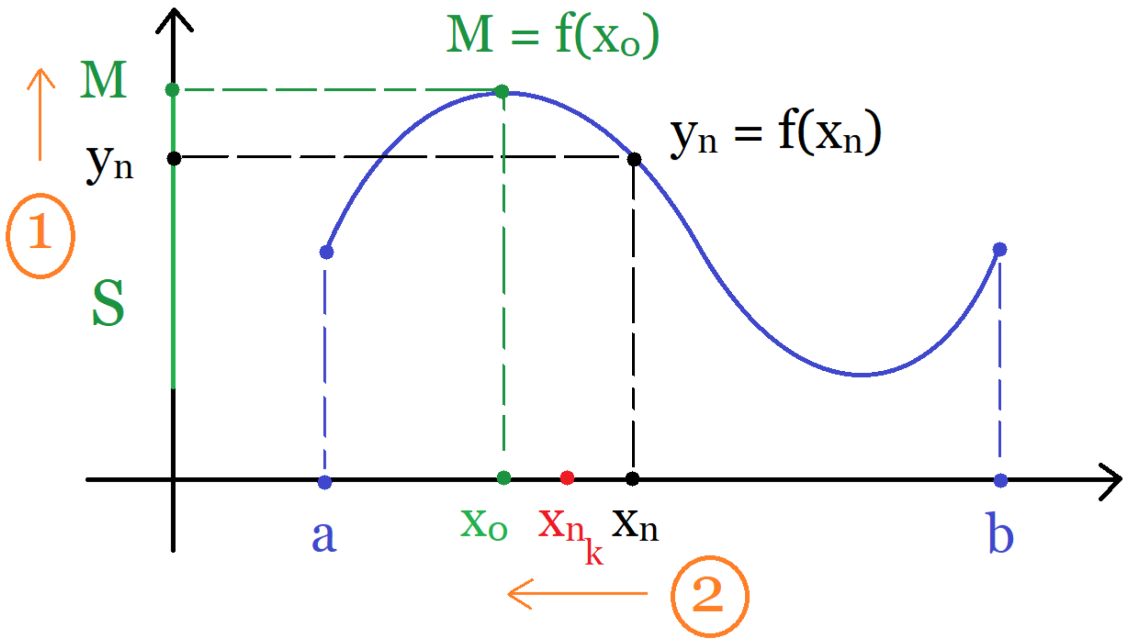
STEP 1: Since f is continuous on $[a, b]$, f is bounded, so consider

$$S = \{f(x) \mid x \in [a, b]\}$$



Then S is bounded (since f is bounded), and therefore S has a least upper bound $\sup(S) =: M$

STEP 2: By the Useful Lemma above, there is a sequence $y_n \in S$ with $y_n \to \sup(S) = M$



Since $y_n \in S$, we get $y_n = f(x_n)$ for some $x_n \in [a, b]$ (by def of S)

Since $x_n \in [a, b]$, (x_n) is bounded. Therefore, by Bolzano-Weierstraß, (x_n) has a convergent subsequence $x_{n_k} \to x_0$ for some $x_0 \in [a, b]$

On the one hand, since $x_{n_k} \to x_0$ and f is continuous, $f(x_{n_k}) \to f(x_0)$

On the other hand, $f(x_n) = y_n \to M$ (by definition of y_n). In particular, the subsequence $f(x_{n_k}) = y_{n_k} \to M$ as well.

Comparing the two limits, we get $f(x_0) = M$

But for all $x \in [a, b]$, we have

$$f(x_0) = M = \sup \{f(x) \mid x \in [a, b]\} \geq f(x)$$

And therefore $f(x_0) \geq f(x)$ for all x, so f has a max at x_0 □

Note: What makes this work is that $[a, b]$ is *compact*. Check out this video if you're interested: Continuity and Compactness

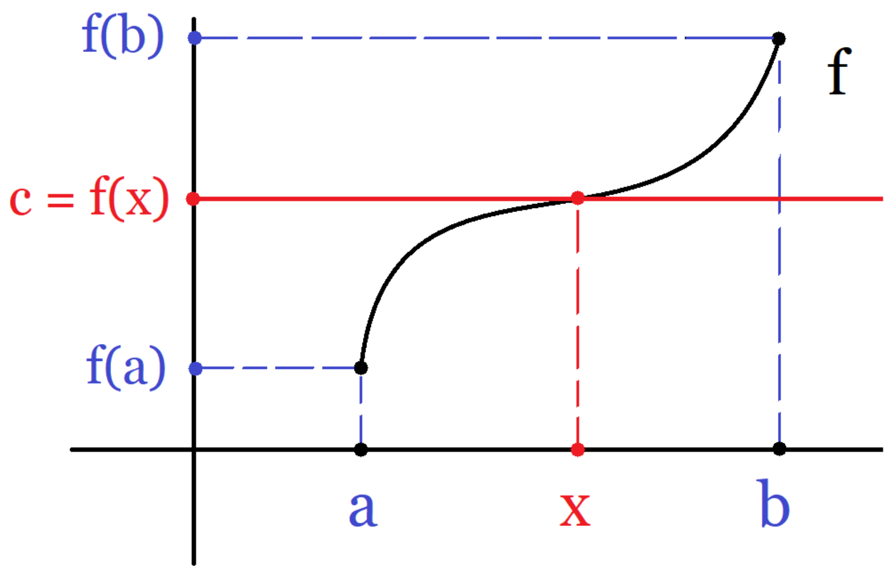
3. The Intermediate Value Theorem

Video: Intermediate Value Theorem

Let's now discuss the second Value Theorem of Calculus: The Intermediate Value Theorem. It says that if f is continuous, then f attains all the values between $f(a)$ and $f(b)$:

Intermediate Value Theorem:

If $f : [a, b] \to \mathbb{R}$ is continuous and if c is any number between $f(a)$ and $f(b)$, then there is some $x \in [a, b]$ such that $f(x) = c$

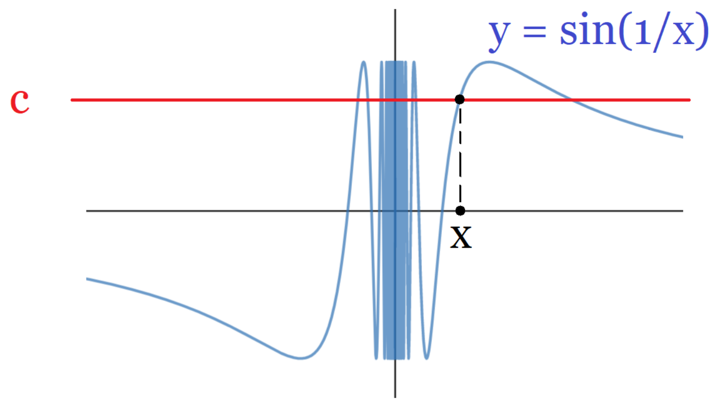


Note: There are functions f that are not continuous, but that satisfy the intermediate value property above.

Example: (see HW)

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Discontinuous at 0 but satisfies the intermediate value property

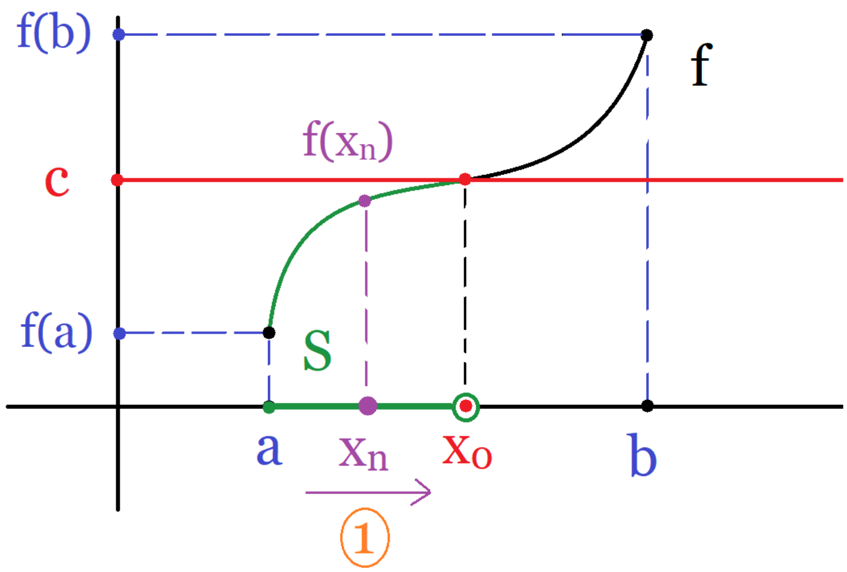


Proof of the Intermediate Value Theorem:

STEP 1: WLOG, assume $f(a) < c < f(b)$ and consider

$$S = \{x \in [a,b] \mid f(x) < c\}$$

Then $S \neq \emptyset$ (because $a \in S$) and S is bounded above by b, therefore S has a least upper bound $\sup(S) =: x_0$



Claim: $f(x_0) = c$

We will do this by showing $f(x_0) \leq c$ and $f(x_0) \geq c$

STEP 2: Show $f(x_0) \leq c$

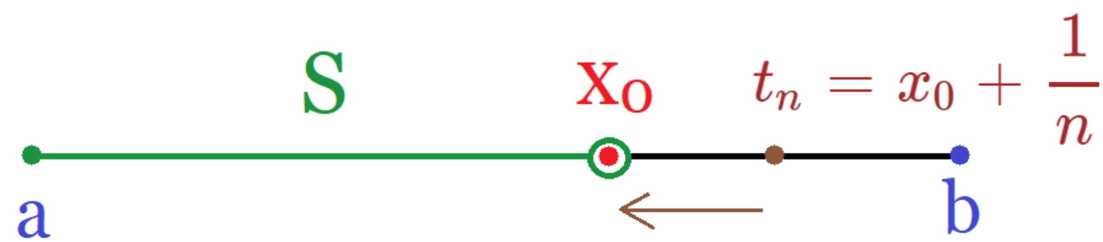
By the Useful Lemma from above, there is a sequence (x_n) in S with $x_n \to x_0$. Therefore, since f is continuous, we get $f(x_n) \to f(x_0)$.

But since $x_n \in S$, by definition of S, we have $f(x_n) < c$, and therefore

$$f(x_0) = \lim_{n\to\infty} f(x_n) \leq c\checkmark$$

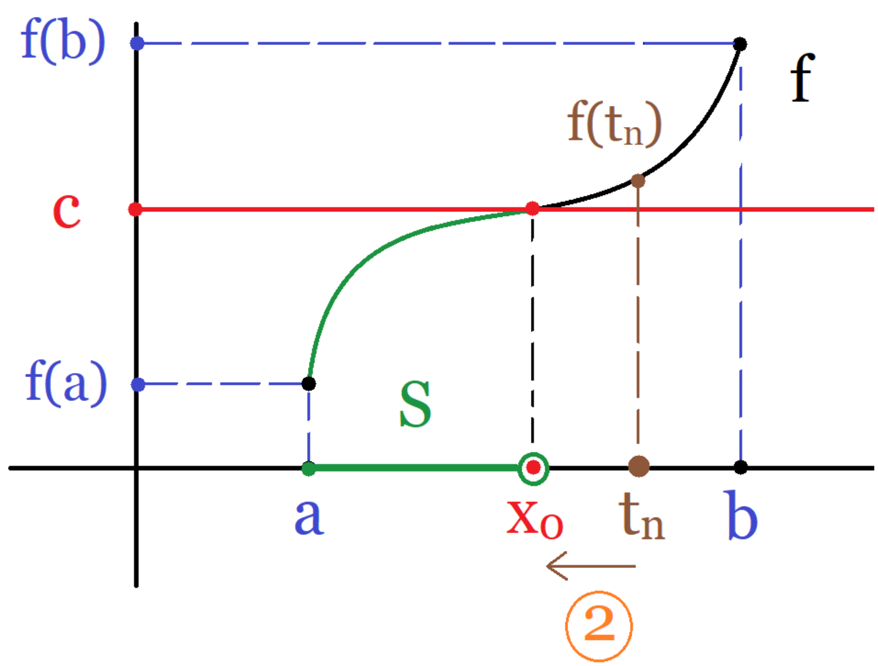
STEP 3: Show $f(x_0) \geq c$

First of all, we have $x_0 \neq b$ because $f(x_0) \leq c$ whereas $f(b) > c$. Hence $x_0 < b$.



Since $x_0 < b$, for n large enough, we have $t_n =: x_0 + \frac{1}{n} < b$

So $t_n \in [a, b]$, $t_n > x_0$, and $t_n \to x_0$.



Since $t_n \to x_0$ and f is continuous, we have $f(t_n) \to f(x_0)$

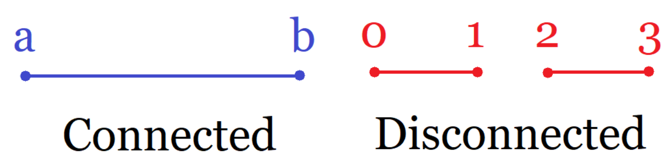
Moreover, since $t_n > x_0$ and $x_0 = \sup(S)$, we must have $t_n \notin S$, meaning that (by definition of S), $f(t_n) \geq c$.

Hence, we have

$$f(x_0) = \lim_{n\to\infty} f(t_n) \geq c\checkmark$$

Combining **STEP 2** and **STEP 3**, we have $f(x_0) = c$ $\square$

Note: The same result holds if you replace $[a, b]$ by any *connected* set. Connected just means that the set just had one piece. For instance $[a, b]$ is connected but $[0, 1] \cup [2, 3]$ is disconnected; it has two pieces.



4. Application 1: Fixed Points

Video: What is a fixed point?

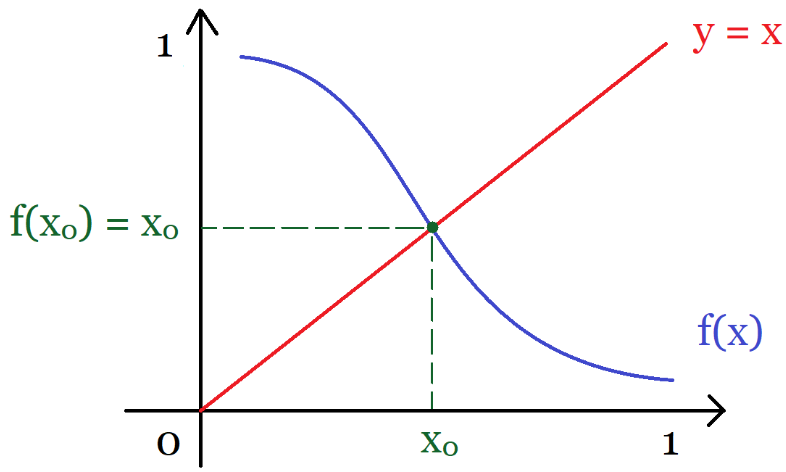
Here are two fun applications of the IVT:

Application 1:

If $f : [0, 1] \to [0, 1]$ is continuous, then f has a **fixed point**: there is some x_0 in $[0, 1]$ such that $f(x_0) = x_0$

(In other words, there is a point x_0 such that, when you apply f to it, then nothing happens)

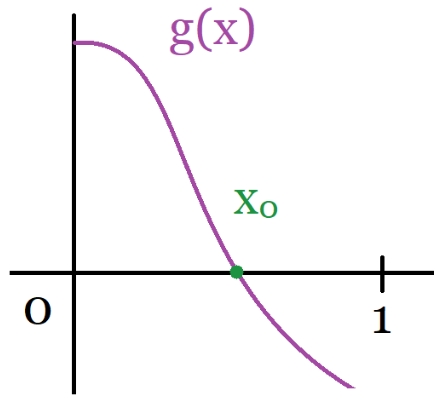
Geometrically, this means that, f must cross the line $y = x$.



Proof: Let $g(x) = f(x) - x$, then g is continuous since f is continuous

Moreover, $g(0) = f(0) - 0 = f(0) \geq 0$

And $g(1) = f(1) - 1 \leq 0$ since $f(x) \leq 1$ for all x.



Therefore, by the IVT with $c = 0$ there is x_0 in $[0, 1]$ such that

$$g(x_0) = 0 \Rightarrow f(x_0) - x_0 = 0 \Rightarrow f(x_0) = x_0 \quad \square$$

Applications: Fixed points are very important in math and beyond. Here are some fun applications of fixed points:

(1) No matter how well you shake a snowglobe, then there is always one snowflake which lands on exactly same position it started!

(2) Take an ordinary map of a country, and suppose that that map is laid out on a table inside that country. There will always be a 'You are Here' point on the map which represents that same point in the country.

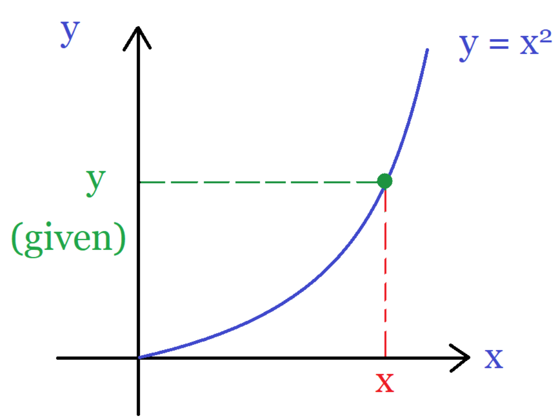
5. Application 2: Square Roots

Video: What is a square root?

As a second application, let's show that $\sqrt{x} = x^{\frac{1}{2}}$ exists

Application 2:

If $y \geq 0$, then there is a unique $x \geq 0$ with $y = x^2$



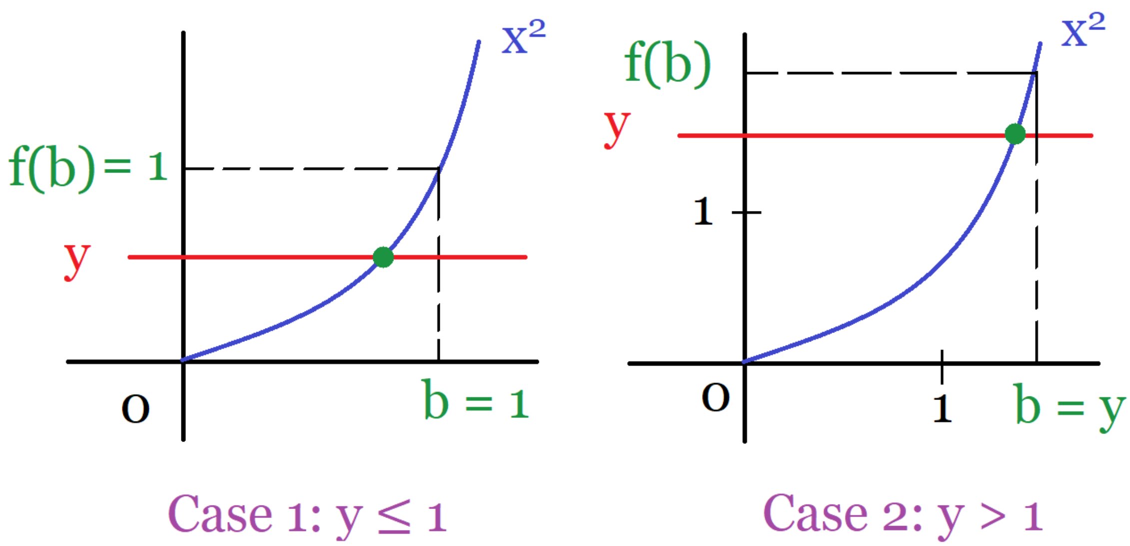
Note: The x as above is called the **square root of** y, $x = \sqrt{y}$

Proof: **Existence:** Let $y \geq 0$ be given and consider $f(x) = x^2$ (continuous).

Then, on the one hand, $f(0) = 0 \leq y$.

On the other hand, there is b such that $f(b) \geq y$ (if $y \leq 1$, let $b = 1$, and if $y > 1$, let $b = y$, see picture below)

Since $f(0) \leq y \leq f(b)$, by the IVT, there is $x \geq 0$ such that $f(x) = y$, that is $x^2 = y$ ✓



Uniqueness: Suppose $a, b \geq 0$ are such that $a^2 = y$ and $b^2 = y$. Then if $a < b$, then we get $a^2 < b^2$, so $y < y \Rightarrow\Leftarrow$ Similar contradiction if $a > b$, and therefore $a = b$ ✓ $\square$

Note: For a more elementary proof that doesn't use the IVT, check this video: Construction of $\sqrt{x}$