LECTURE 17: MAX AND MIN (II), LAGRANGE MULTIPLIERS (I)

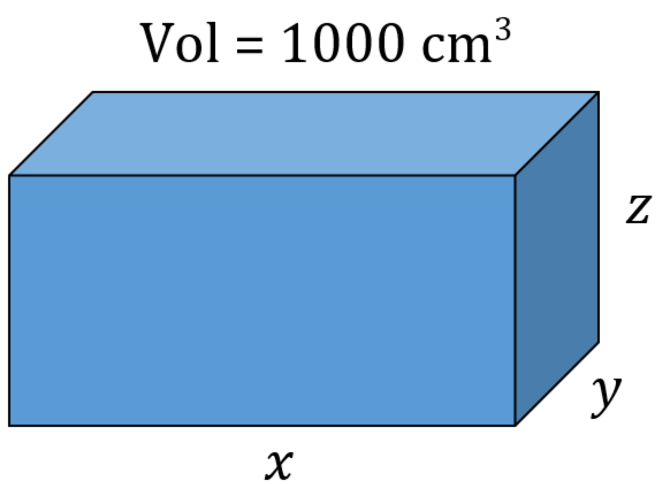
1. IKEA Problem

Video: IKEA Problem

Hello everyone and welcome to IKEA! Today, in addition to eating delicious meatballs, we would like to design a special box named Malmök

Example 1:

Find the smallest surface area of a box whose volume is 1000 cm^3



STEP 1: Find $f(x, y)$

Date: Wednesday, October 6, 2021.

The surface area of the box with length x, width y and height z is:

$$S = 2xy + 2yz + 2xz$$

We want to write this in terms of x and y only. However

$$V = xyz = 1000 \Rightarrow z = \frac{1000}{xy}$$

$$\text{Therefore } S = 2xy + 2y\left(\frac{1000}{xy}\right) + 2x\left(\frac{1000}{xy}\right)$$

$$f(x,y) = 2xy + \frac{2000}{x} + \frac{2000}{y}$$

STEP 2:

$$f_x = 2y - \frac{2000}{x^2} = 0 \Rightarrow 2y = \frac{2000}{x^2} \Rightarrow y = \frac{1000}{x^2}$$
$$f_y = 2x - \frac{2000}{y^2} = 0 \Rightarrow 2x = \frac{2000}{y^2} \Rightarrow x = \frac{1000}{y^2}$$

However, remember that $y = \frac{1000}{x^2}$, therefore:

$$x = \frac{1000}{y^2} = \frac{1000}{\left(\frac{1000}{x^2}\right)^2} = \left(\frac{1000}{1000^2}\right)x^4 = \frac{x^4}{1000}$$

But assuming $x \neq 0$, we get:

$$x = \frac{x^4}{1000} \Rightarrow 1 = \frac{x^3}{1000} \Rightarrow x^3 = 1000 \Rightarrow x = \sqrt[3]{1000} \Rightarrow x = 10$$

And then we get

$$y = \frac{1000}{x^2} = \frac{1000}{10^2} = 10$$

Therefore the only critical point is $(10, 10)$

STEP 3: $f_x = 2y - \frac{2000}{x^2} = 2y - 2000x^{-2}$ and $f_y = 2x - \frac{2000}{y^2}$

$f_{xx} = -2000(-2)x^{-3} = \frac{4000}{x^3}$, $f_{xy} = f_{yx} = 2$ and $f_{yy} = \frac{4000}{y^3}$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} \frac{4000}{x^3} & 2 \\ 2 & \frac{4000}{y^3} \end{vmatrix}$$

$$D(10,10) = \begin{vmatrix} \frac{4000}{1000} & 2 \\ 2 & \frac{4000}{1000} \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} = 12 > 0$$

And $f_{xx}(10, 10) = 4 > 0$, therefore f has a local minimum at $(10, 10)$

STEP 4: Finally, to find the smallest surface area, use

$$z = \frac{1000}{xy} = \frac{1000}{10 \times 10} = 10$$

Hence $x = 10, y = 10, z = 10$ and

$S = 2xy + 2yz + 2xz = 2(10)(10) + 2(10)(10) + 2(10)(10) = 600$ cm ^{2}

Note: The optimal box here is a cube! Isn't it pretty how nature balances itself out like that? ☺

Note: For more practice, try the following related problem:

Extra Practice:

Find the **largest** possible volume of a box whose surface area is 600 cm^2 (In that case you should also find that it's a cube)

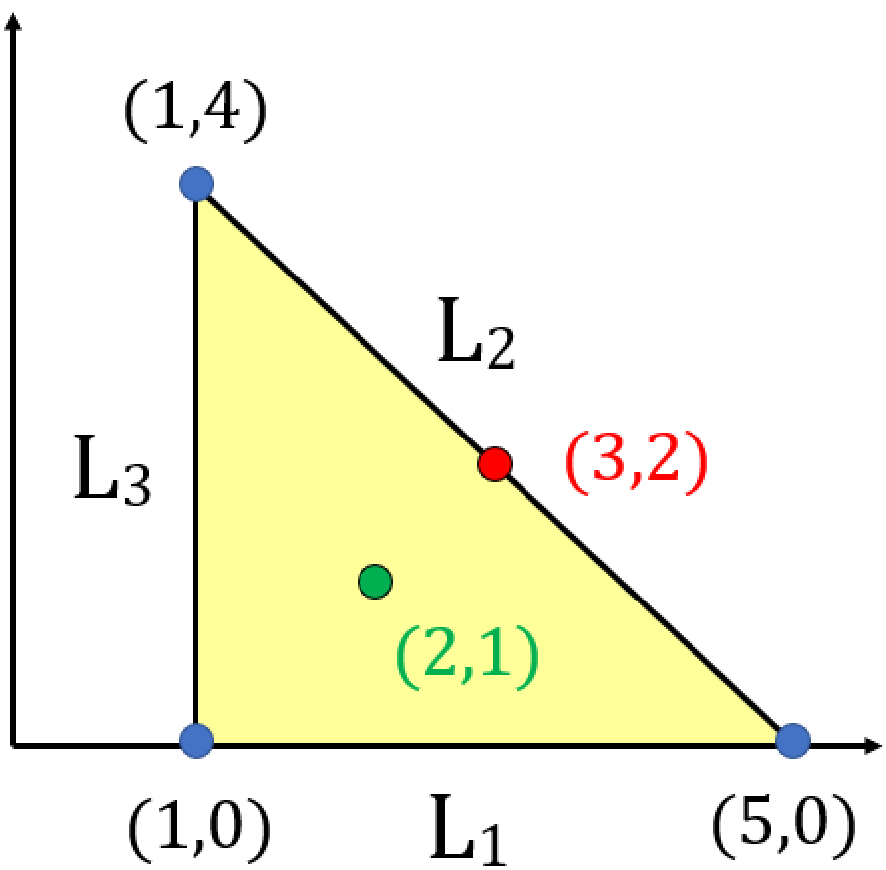
2. Absolute Max and Min

Sometimes you also want to find the **absolute** max/min of a function

Example 2:

Find the absolute max/min of the following function on the triangle with vertices $(1,0),(5,0),(1,4)$

$$f(x,y) = 2 + xy - x - 2y$$



Pro: Don't need second derivatives

Con: Need to check "endpoints"

Idea: Find all the potential candidates for max/min and then compare them at the end.

STEP 1: Critical Points

$$\begin{aligned} f_x =& y - 1 = 0 \Rightarrow y = 1 \\ f_y =& x - 2 = 0 \Rightarrow x = 2 \end{aligned}$$

Hence the only critical point is $\boxed{(2,1)}$, which is inside the triangle (if it's not, then you ignore it)

Note: For *absolute* max, don't need to take second derivatives!

STEP 2: Endpoints

Here the triangle has 3 endpoints: $\boxed{(1,0),(5,0),(1,4)}$

STEP 3: Find the critical points on each line segment.

Case 1: On L_1, we have $y = 0$, so

$$f(x,y) = f(x,0) = 1 + x(0) - x - 2(0) = 2 - x$$

$(2-x)' = -1 \neq 0$, so there are no critical points on L_1

Case 2: On L_2, we have $y = 5 - x$ (line connecting $(1,4)$ and $(5,0)$):

$$\begin{aligned} f(x,y) =& f(x, 5-x) \\ =& 2 + x(5-x) - x - 2(5-x) \\ =& 2 + 5x - x^2 - x - 10 + 2x \\ =& -x^2 + 6x - 8 \end{aligned}$$

$$\left(-x^2+6x-8\right)' = -2x+6 = 0 \Rightarrow x = 3$$

But if $x = 3$ then $y = 5 - x = 5 - 3 = 2$ so our critical point is $\boxed{(3,2)}$

Case 3: On L_3, we have $x = 1$ and

$$f(x,y) = f(1,y) = 2 + (1)y - 1 - 2y = 1 - y$$

$(1-y)' = -1 \neq 0$, so there are no critical points on L_3

STEP 4: Our Candidates

$(1,2)$	$f(2,1) = 0$
$(1,0)$	$f(1,0) = 1$
$(5,0)$	$f(5,0) = -3$
$(1,4)$	$f(1,4) = -3$
$(3,2)$	$f(3,2) = 1$

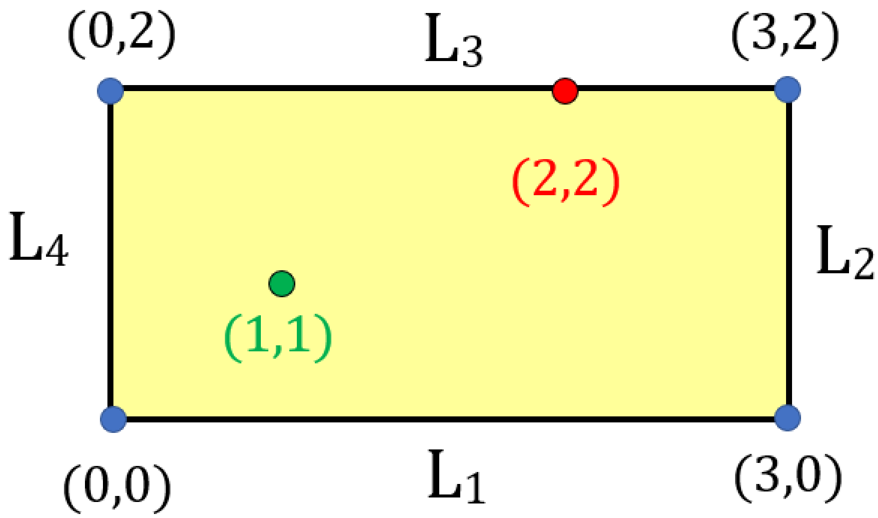
The **absolute** max (biggest value) is $f(1,0) = f(3,2) = 1$

The **absolute** min (smallest value) is $f(5,0) = f(1,4) = -3$

Example 3: (Extra Practice)

Find the **absolute** max/min of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle

$$R = \{(x,y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$$



STEP 1: Critical Points

$$f_x = 2x - 2y = 0 \Rightarrow 2x = 2y \Rightarrow x = y$$
$$f_y = -2x + 2 = 0 \Rightarrow 2x = 2 \Rightarrow x = 1$$

Hence $x = 1$ and $y = x = 1$ so the critical point is $\boxed{(1,1)}$, which is in R

STEP 2: Endpoints

There are four endpoints: $\boxed{(0,0),(3,0),(3,2),(0,2)}$

STEP 3: Critical points on sides

Notice the boundary (sides) of R consists of 4 line segments L_1, L_2, L_3, L_4

Case 1: On L_1, we have $y = 0$, so

$$f(x,y) = f(x,0) = x^2 - 2x(0) + 2(0) = x^2$$

$\left(x^2\right)' = 2x = 0$ which gives $x = 0$ (and $y = 0$), which gives $(0, 0)$, but which we have already considered in **STEP 2**

Case 2: On L_2, we have $x = 3$ and

$$f(x, y) = f(3, y) = 3^2 - 2(3)y + 2y = 9 - 4y$$

But $(9 - 4y)' = -4 \neq 0$, so no critical points

Case 3: On L_3, we have $y = 2$ and

$$f(x, y) = f(x, 2) = x^2 - 4x + 4$$

$\left(x^2 - 4x + 4\right)' = 2x - 4 = 0$ which gives $x = 2$ (and $y = 2$), which gives the critical point $\boxed{(2, 2)}$

Case 4: On L_4, we have $x = 0$ and

$$f(x, y) = f(0, y) = 2y$$

$(2y)' = 2 \neq 0$, so no critical points on L_4

STEP 4: Our Candidates

$(1, 1)$	$f(1, 1) = 1$
$(0, 0)$	$f(0, 0) = 0$
$(3, 0)$	$f(3, 0) = 9$
$(3, 2)$	$f(3, 2) = 1$
$(0, 2)$	$f(0, 2) = 4$
$(2, 2)$	$f(2, 2) = 0$

The **absolute** max is $f(3, 0) = 9$

The **absolute** min is $f(0, 0) = f(2, 2) = 0$

3. Lagrange Multipliers

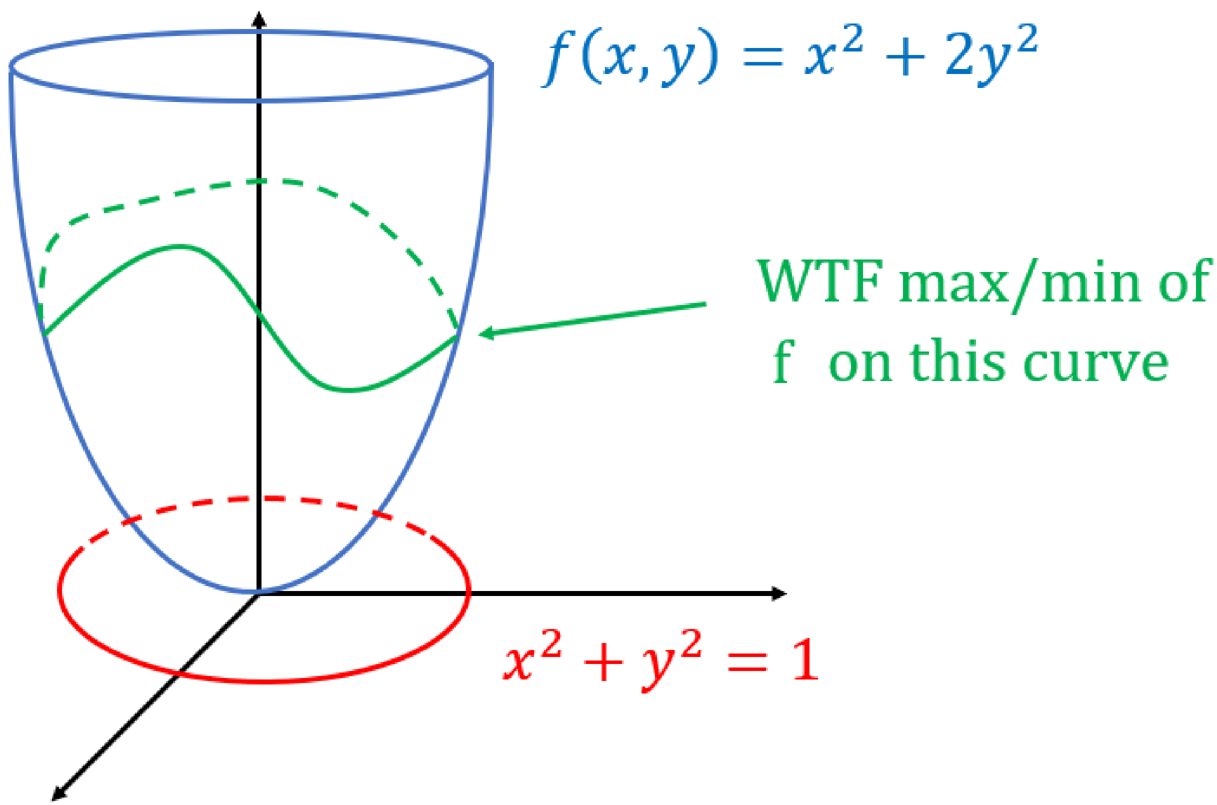
What if our region is more complicated? Then we have to use the power of Multivariable Calculus ☺

Example 4: (Motivation)

Find the absolute max/min of the following function on the circle $x^2 + y^2 = 1$ (constraint)

$$f(x, y) = x^2 + 2y^2$$

Picture:



Think of it as follows: Suppose you're walking in a circle on a mountain, then what is your biggest and smallest height?

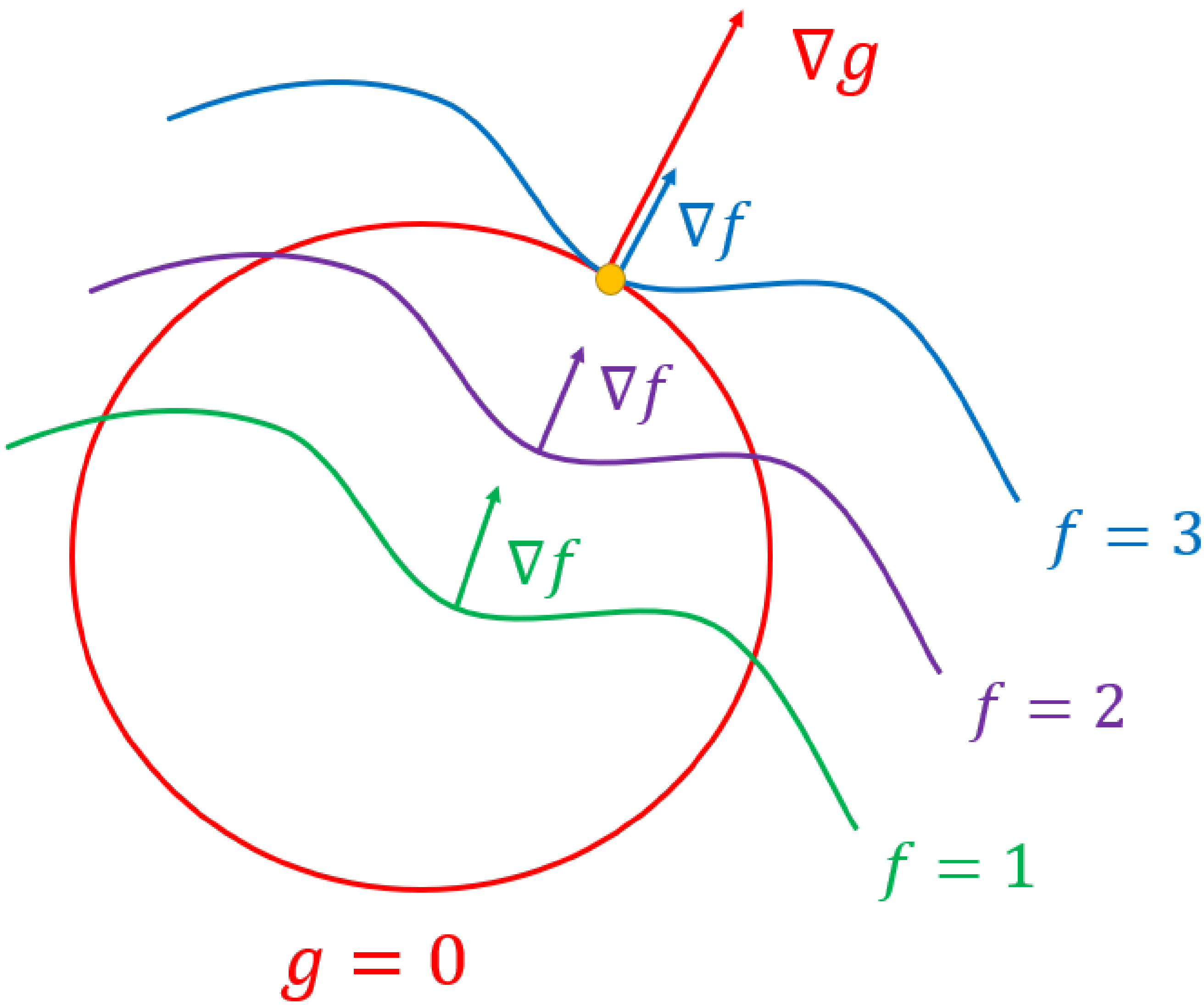
In fact, since we're talking about mountains, the solution to this problem are level curves!

Notation: $x^2 + y^2 = 1 \Rightarrow \underbrace{x^2 + y^2 - 1}_{g(x,y)} = 0 \Rightarrow g = 0$

Recall:

∇g is perpendicular to level curves of g

Now think of $g = 0$ as fixed, and draw level curves of f until you reach the curve of the highest point:



Important Observation:

At the max/min, ∇f and ∇g are parallel, that is $\nabla f = c\nabla g$ for some c

Which ultimately leads to:

Lagrange Equation:

$$\nabla f = \lambda \nabla g$$

Here λ is called the **Lagrange multiplier**. It's a useful tool to find absolute max/min of a function given a constraint.

4. Finding Absolute Max/Min

Let's now use the Lagrange equation to solve our original problem:

Example 5:

Find the absolute max/min of $f(x, y) = x^2 + 2y^2$ on $x^2 + y^2 = 1$

$g(x, y) = x^2 + y^2 - 1$

STEP 1: Lagrange Equation: $\nabla f = \lambda \nabla g$:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} 2x = \lambda(2x) \\ 4y = \lambda(2y) \\ x^2 + y^2 = 1 \end{cases}$$

STEP 2: Collect Points

Tips:

(1) Remember that you have to solve for x and y, not λ. Here λ is just a helper (like Yoshi in Super Mario)

(2) Use the constraint $x^2 + y^2 = 1$ **last**

$$\begin{cases} 2x = \lambda(2x) \Rightarrow x - \lambda x = 0 \Rightarrow x(1-\lambda) = 0 \Rightarrow x = 0 \text{ or } \lambda = 1 \\ 4y = \lambda(2y) \Rightarrow 2y - \lambda y = 0 \Rightarrow y(2-\lambda) = 0 \Rightarrow y = 0 \text{ or } \lambda = 2 \end{cases}$$

Case 1: $x = 0$ and $y = 0$, but then $x^2 + y^2 \neq 1$ ✗

Case 2: $x = 0$ and $\lambda = 2$, then

$$x^2 + y^2 = 1 \Rightarrow 0^2 + y^2 = 1 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$$

Which gives the candidates $\boxed{(0,1),(0,-1)}$

Case 3: $\lambda = 1$ and $y = 0$, then

$$x^2 + y^2 = 1 \Rightarrow x^2 + 0^2 = 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

Which gives the candidates $\boxed{(1,0),(-1,0)}$

Case 4: $\lambda = 1$ and $\lambda = 2$ Impossible ✗

STEP 3: Compare

$(0,-1)$	$f(0,-1) = 2$
$(0,1)$	$f(0,1) = 2$
$(1,0)$	$f(1,0) = 1$
$(-1,0)$	$f(-1,0) = 1$

The **absolute** max is $f(0,1) = f(0,-1) = 2$

The **absolute** min is $f(1,0) = f(-1,0) = 1$

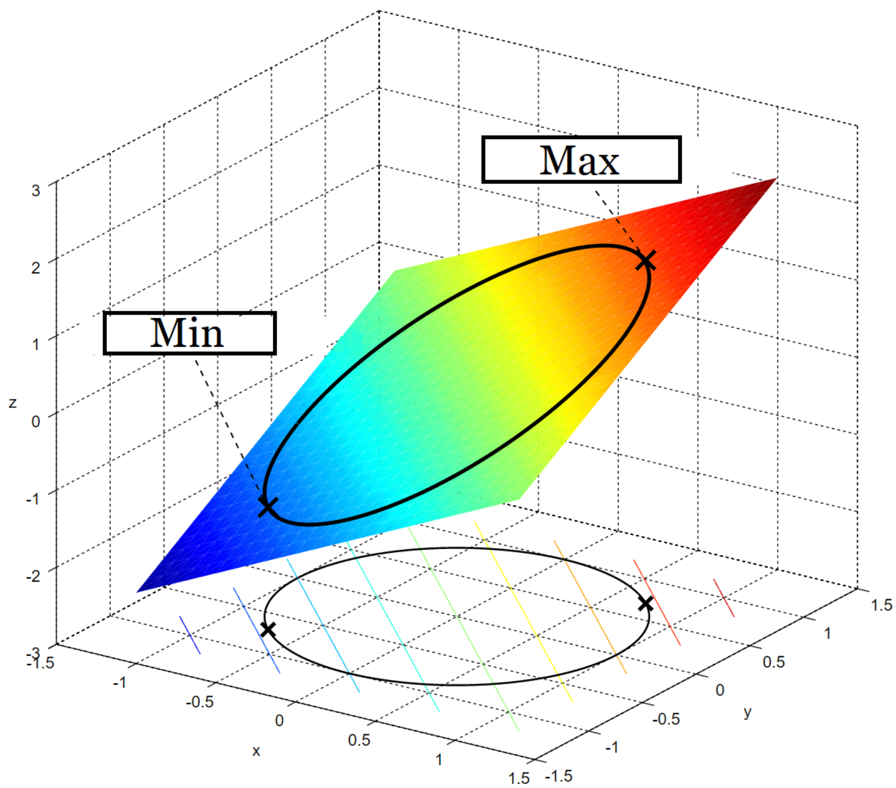
5. More Practice

There are essentially two types of Lagrange problems: Either you solve for x and y in cases, or you solve for x and y in terms of λ, as the following example shows

Example 6: (extra practice)

Find the absolute max/min of $f(x, y) = 8x + 10y$ on $x^2 + y^2 = 41$

Notice $z = 8x + 10y$ is a plane, so we have to find the biggest and smallest value of a plane, given that we're walking on a circle:[1]



[1]The picture was adapted from Wikipedia

$f(x, y) = 8x + 10y, \quad g(x, y) = x^2 + y^2 - 41$

STEP 1: Lagrange Equation

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ x^2 + y^2 = 41 \end{cases} \Rightarrow \begin{cases} 8 = \lambda(2x) \\ 10 = \lambda(2y) \\ x^2 + y^2 = 41 \end{cases}$$

STEP 2: Collect Points

This time it's useful to solve for λ in terms of x and y. Notice that $x \neq 0$ and $y \neq 0$ are both nonzero, otherwise the above equations wouldn't hold

$$\begin{cases} 8 = \lambda(2x) \Rightarrow x = \dfrac{8}{2\lambda} = \dfrac{4}{\lambda} \\ 10 = \lambda(2y) \Rightarrow y = \dfrac{10}{2\lambda} = \dfrac{5}{\lambda} \end{cases}$$

Therefore the constraint $x^2 + y^2 = 41$ becomes:

$$\begin{aligned} \left(\frac{4}{\lambda}\right)^2 + \left(\frac{5}{\lambda}\right)^2 &= 41 \\ \frac{16}{\lambda^2} + \frac{25}{\lambda^2} &= 41 \\ \frac{41}{\lambda^2} &= 41 \\ \frac{1}{\lambda^2} &= 1 \\ \lambda^2 &= 1 \\ \lambda &= \pm 1 \end{aligned}$$

Case 1: $\lambda = 1$, but then:

$$x = \frac{4}{\lambda} = \frac{4}{1} = 4, \qquad y = \frac{5}{\lambda} = 5$$

Which gives the point $\boxed{(4,5)}$

Case 2: $\lambda = -1$, but then:

$$x = \frac{4}{\lambda} = \frac{4}{-1} = -4, \qquad y = \frac{5}{\lambda} = -5$$

Which gives the point $\boxed{(-4,-5)}$

STEP 3: Compare

$(4,5)$	$f(4,5) = 82$
$(-4,-5)$	$f(-4,-5) = -82$

The **absolute** max is $f(4,5) = 82$

The **absolute** min is $f(-4,-5) = -82$