LECTURE 18: UNIFORM CONTINUITY (I)

1. Image of an interval

Video: Image of an interval

Because the Intermediate Value Theorem, it is interesting to figure out what happens when you apply a function to an interval.

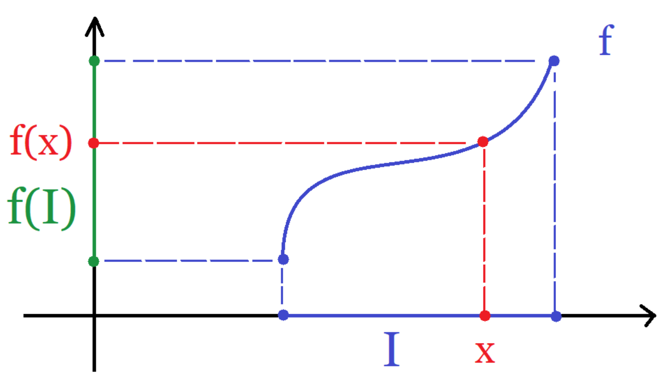
Notation:

I is an interval, such as $I = (0,1)$ or $[1,2)$ or $[2,3]$ or $(3,\infty)$ or even $\mathbb{R}$

Definition:

If I is an interval then the **image of f of I** (or the range of f) is

$$f(I) = \{f(x) \mid x \in I\}$$

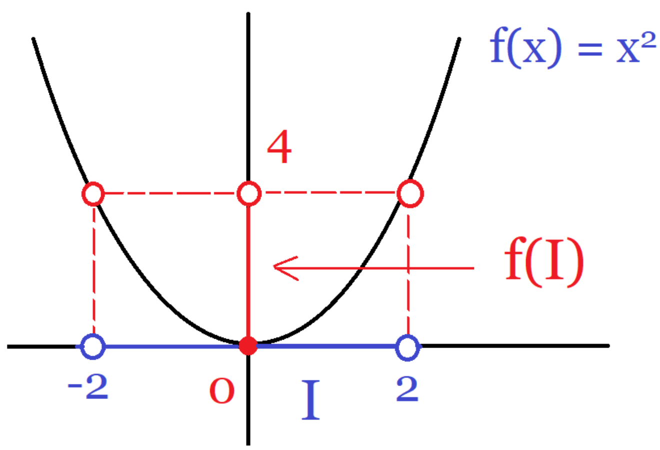


Date: Thursday, October 28, 2021.

Example 1:

If $f(x) = x^2$ and $I = (-2, 2)$ then

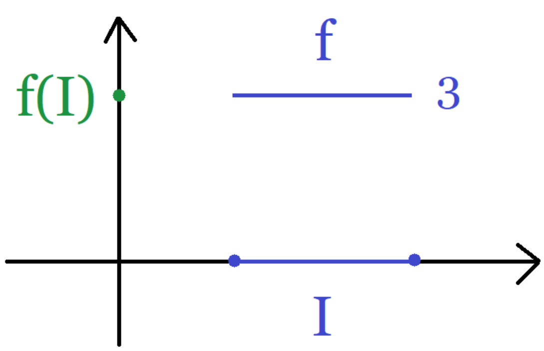
$$f(I) = \left\{x^2 \mid x \in (-2, 2)\right\} = [0, 4)$$



Example 2:

If $f(x) = 3$ and I is any nonempty interval, then

$$f(I) = \{3\}$$



Notice that in each of those examples, $f(I)$ is either a point or an interval. It turns out this is always true:

Fact:

If f is continuous, then $f(I)$ is an interval (or a single point)

2. Continuous Functions are Monotonic

Video: Continuity and Monotonicity

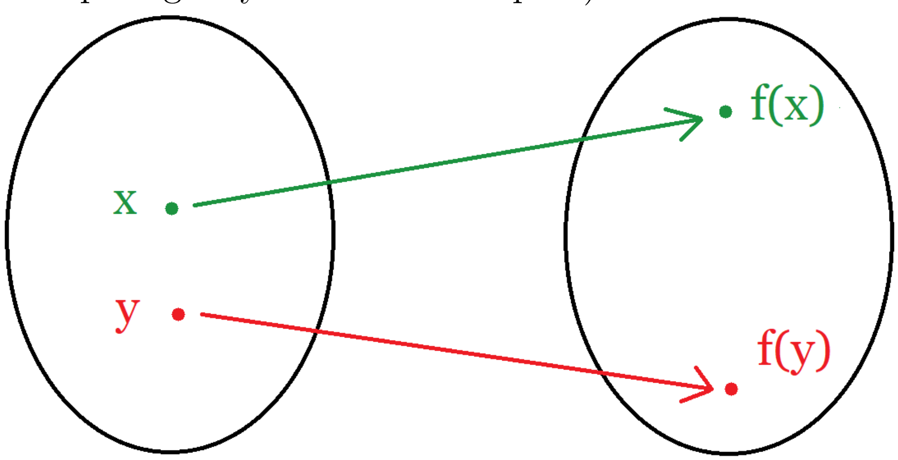
Here's another truly amazing fact about continuous functions: Continuous one-to-one functions must be either increasing or decreasing!

Definition:

f is **one-to-one** if and only if

$$x \neq y \Rightarrow f(x) \neq f(y)$$

(Different inputs give you different outputs)



Definition:

f is (strictly) **increasing** if $a < b \Rightarrow f(a) < f(b)$

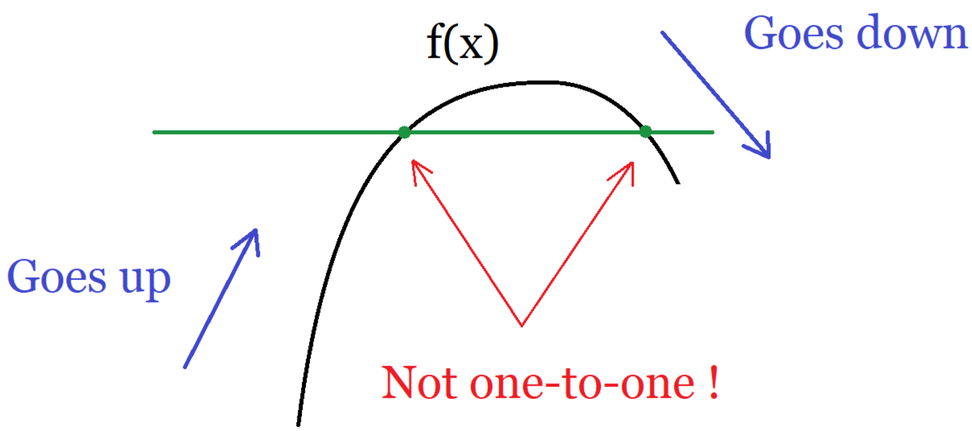
f is (strictly) **decreasing** if $a < b \Rightarrow f(a) > f(b)$

In either case, f is (strictly) **monotonic**

Fact:

If $f : I \to \mathbb{R}$ is one-to-one and continuous, then f must be monotonic

Intuitively, this makes sense: Suppose f is not monotonic. Then, f goes up and down (or down and up) and cannot be one-to-one.



There's a fun application of this on the homework.

3. f^{-1} IS CONTINUOUS

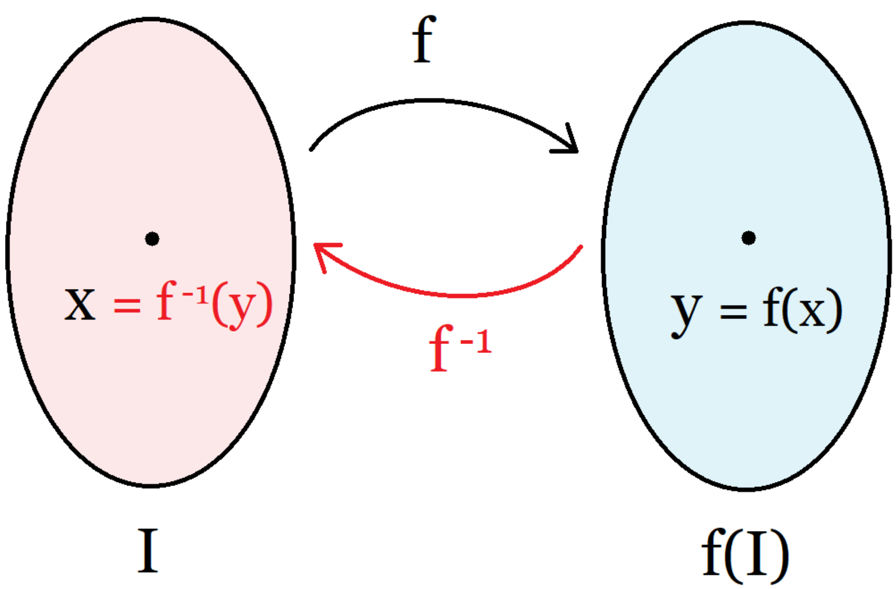
Video: f^{-1} is continuous

Finally, let's state the incredible fact that if a real-valued f is continuous, then f^{-1} is continuous as well. This explains the fact why $\sqrt{x}, \tan^{-1}(x)$, or even $\ln(x)$ are continuous.

Definition:

If $f : I \to f(I)$ is one-to-one, $f^{-1} : f(I) \to I$ is defined by

$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$



In other words, if f as a flight from x to y, then f^{-1} is the return flight from y to x. f^{-1} undoes whatever f does.

Theorem:

If $f : I \to f(I)$ is one-to-one and continuous, then $f^{-1} : f(I) \to I$ is continuous as well

4. Uniform Continuity

Video: Uniform Continuity

Now let's talk about a new and improved version of continuity, called **uniform continuity**

Recall:

f is **continuous at** x_0 if for all $\epsilon > 0$ there is $\delta > 0$ such that for all x, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$

Important Observation: δ may or may not depend on x_0.

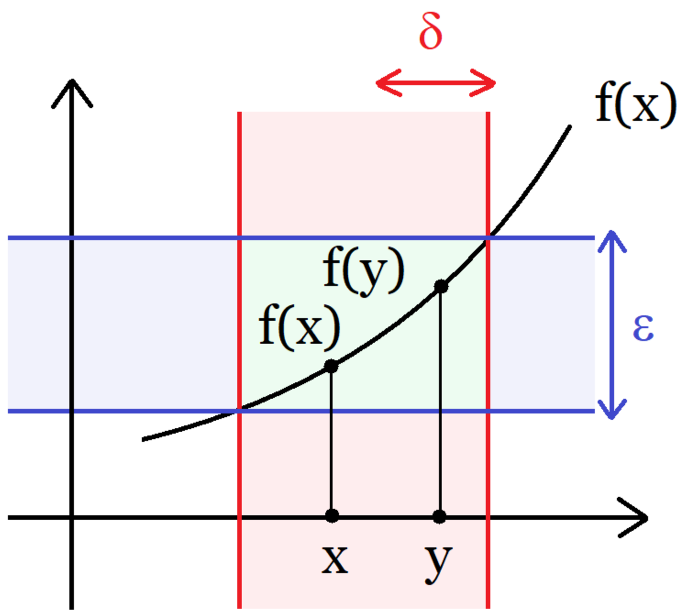
Example 1: If $f(x) = 4x + 3$, then $\delta = \frac{\epsilon}{4}$

Example 2: If $f(x) = 2x^2 + 1$, then $\delta = \min\left\{1, \frac{\epsilon}{2(2|x_0|+1)}\right\}$, which depends on x_0

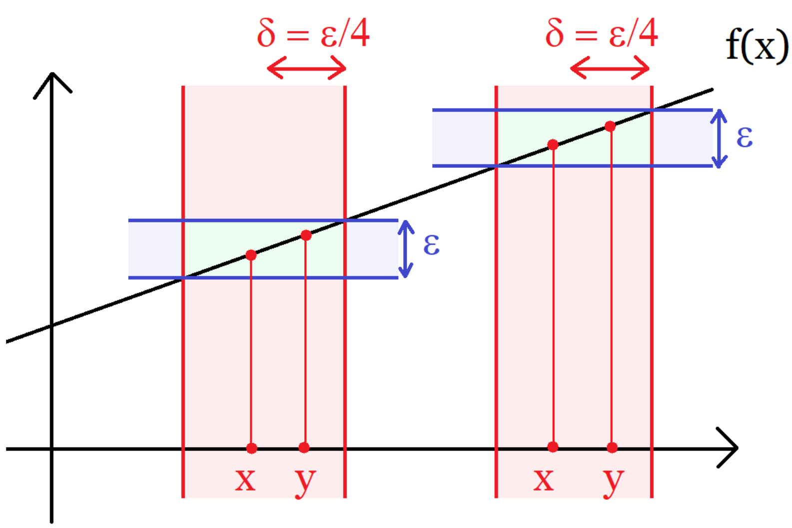
Upshot: *If* δ does not depend on x_0, then f is called **uniformly continuous**:

Definition:

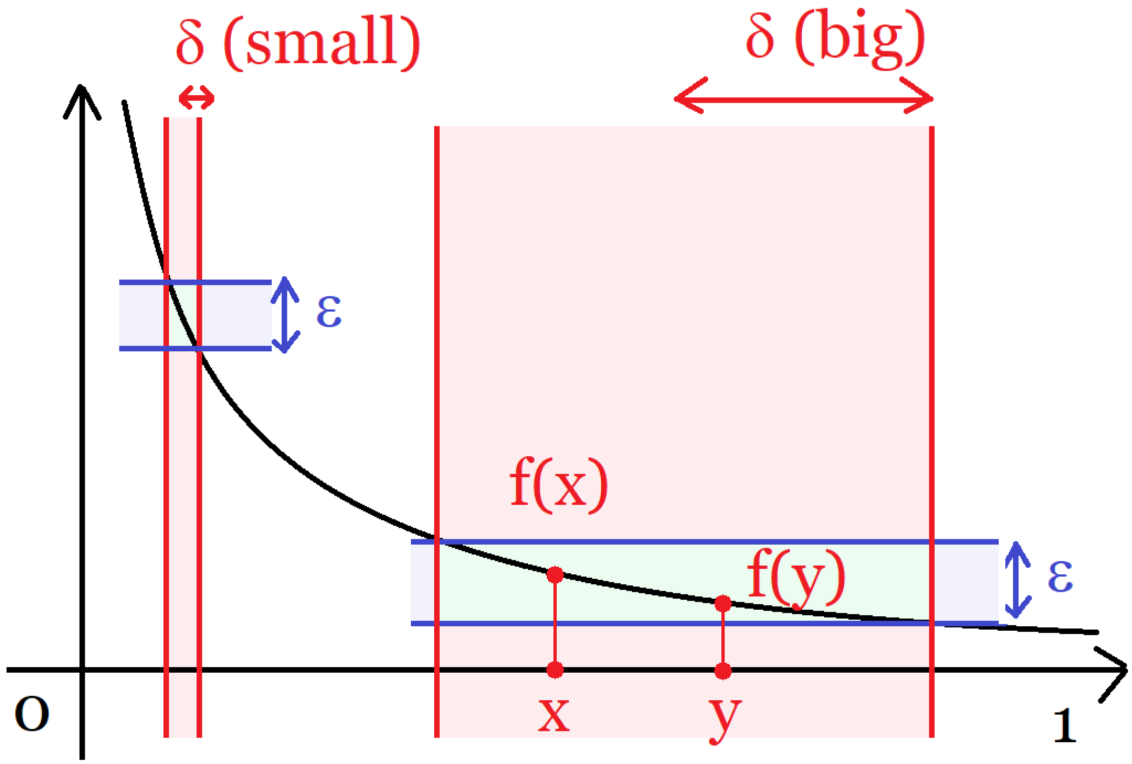
f is **uniformly continuous** on a set S if for all $\epsilon > 0$ there is $\delta > 0$ such that, for all $x, y \in S$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$



Even though the two definitions look identical, the main difference is that in uniform continuity, δ does not depend on x or y: There is a universal δ that works for *all* x and y.



Non-Example: $f(x) = \frac{1}{x}$ is not uniformly continuous on $S = (0, 1)$ because δ *depends* on where you are: Near 1, δ doesn't need to be small in order to have $|f(x) - f(y)| < \epsilon$. But near 0, then δ needs to be *extremely* small in order to guarantee that $|f(x) - f(y)| < \epsilon$:



Careful: The set S matters. For example $f(x) = \frac{1}{x}$ is uniformly continuous on $[2, \infty)$ but $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$ (see Examples below)

5. Example 1: The Basics

Video: Example 1: The Basics

Example 1:

Show $f(x) = x^2$ is uniformly continuous on $[-1, 3]$

Show: for all $\epsilon > 0$ there is $\delta > 0$ such that for all $x, y \in [-1, 3]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$

STEP 1: Scratchwork

$$|f(x) - f(y)| = \left|x^2 - y^2\right| = |x - y|\,|x + y| \leq |x - y|\,(|x| + |y|)$$

Note: The $|x - y|$ term is *good*, so we need to control the $|x| + |y|$ term.

Since $x, y \in [-1, 3]$ we have $|x| \leq 3$ and $|y| \leq 3$, and therefore $|x| + |y| \leq 3 + 3 = 6$

$$\text{Hence } \; |x - y|\underbrace{(|x| + |y|)}_{6} \leq 6\,|x - y| < \epsilon \Rightarrow |x - y| < \frac{\epsilon}{6}$$

Which suggests to let $\delta = \frac{\epsilon}{6}$ (independent of x and y)

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, and let $\delta = \frac{\epsilon}{6}$

Then if $x, y \in [-1, 3]$, then $|x| \leq 3$ and $|y| \leq 3$ and therefore

$$\begin{aligned}
|f(x) - f(y)| = |x - y|\,|x + y| &\leq |x - y|\,(|x| + |y|) \\
&\leq |x - y|\,(3 + 3) = 6\,|x - y| \\
&< 6\left(\frac{\epsilon}{6}\right) \\
&= \epsilon \checkmark
\end{aligned}$$

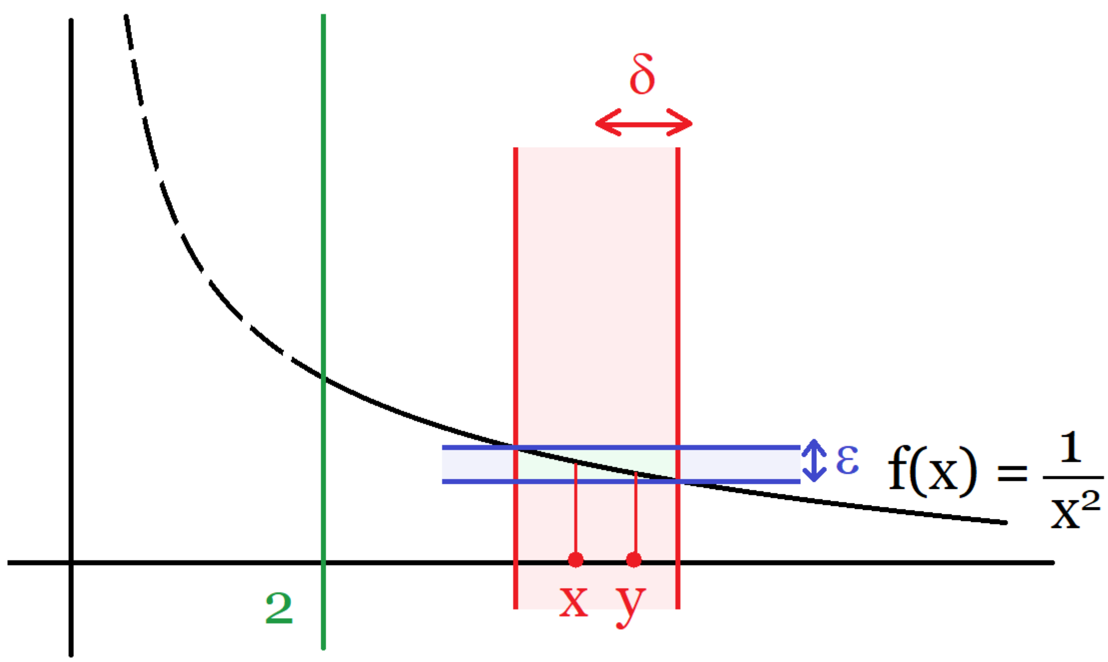
Hence f is uniformly continuous on $[-1, 3]$ $\square$

6. Example 2: $\frac{1}{x^2}$ is uniformly continuous

Video: Example 2: $\frac{1}{x^2}$

Example 2:

Show $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[2, \infty)$



Show: for all $\epsilon > 0$ there is $\delta > 0$ such that for all $x, y \in [2, \infty)$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$

STEP 1: **Scratchwork**

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2y^2}\right| = \frac{|y-x|\,|y+x|}{x^2y^2} = |y-x|\left(\frac{y+x}{x^2y^2}\right)$$

(In the last line, we used $x, y \geq 2$, so $x, y > 0$ and therefore $y + x > 0$)

Here we just need to make the $\frac{y+x}{x^2y^2}$ term constant

$$\text{However: } \frac{y+x}{x^2y^2} = \frac{y}{x^2y^2} + \frac{x}{x^2y^2} = \frac{1}{x^2\,(y)} + \frac{1}{x\,(y^2)}$$

But since $x, y \geq 2$, we have $\frac{1}{x} \leq \frac{1}{2}$ and $\frac{1}{y} \leq \frac{1}{2}$ and therefore

$$\frac{1}{x^2\,(y)} + \frac{1}{x\,(y^2)} \leq \frac{1}{2^2(2)} + \frac{1}{2(2^2)} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

Hence we get

$$|y-x|\left(\frac{y+x}{x^2y^2}\right) \leq |y-x|\left(\frac{1}{4}\right) = \frac{|y-x|}{4} < \epsilon \Rightarrow |x-y| < 4\epsilon$$

Which suggests to let $\delta = 4\epsilon$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, and let $\delta = 4\epsilon$.

Then if $x, y \in [2, \infty)$, then $\frac{1}{x} \leq \frac{1}{2}$ and $\frac{1}{y} \leq \frac{1}{2}$ and therefore

$$\begin{aligned}|f(x)-f(y)| = |y-x|\left(\frac{x+y}{x^2y^2}\right) &\leq |y-x|\left(\frac{1}{x(y^2)}+\frac{1}{x^2(y)}\right)\\ &\leq |y-x|\left(\frac{1}{8}+\frac{1}{8}\right)\\ &= \frac{|y-x|}{4}\\ &< \frac{4\epsilon}{4} = \epsilon \checkmark\end{aligned}$$

Hence f is uniformly continuous on $[2,\infty)$ □

Let's now discuss some neat properties of uniform continuity.

7. Uniform Continuity on $[a,b]$

Video: Uniform Continuity on $[a,b]$

First, let's prove the (unbelievable) fact that continuous functions on $[a,b]$ are in fact uniformly continuous

Fact:

If $f:[a,b]\to\mathbb{R}$ is continuous, then f is uniformly continuous on $[a,b]$

This, for example, gives us a 2 second way of doing Example 1: x^2 continuous on $[-1,3]$, so it is automatically uniformly continuous

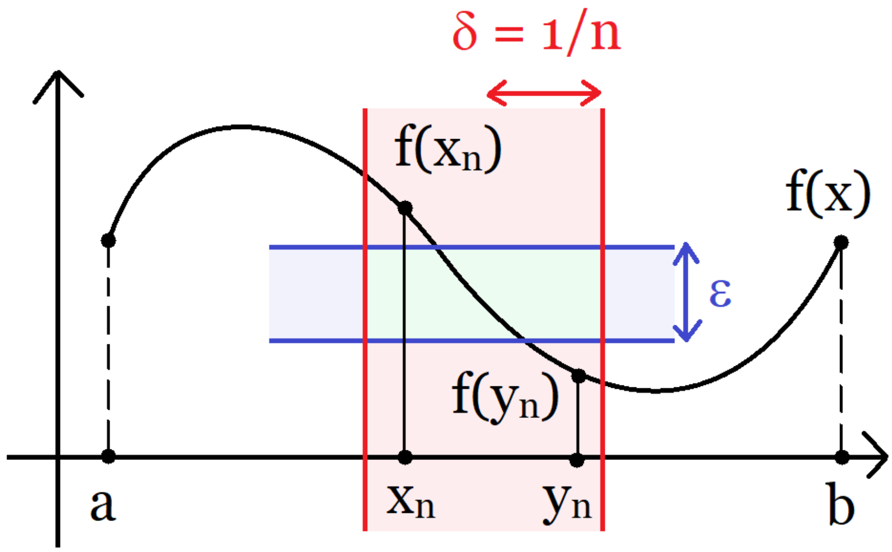
Note: This theorem is **NOT** true for open intervals (a, b) or infinite intervals like $[2, \infty)$, but we can replace $[a, b]$ by any *compact* set.

Proof:

STEP 1: Suppose not, that is f is continuous but not uniformly continuous on $[a, b]$

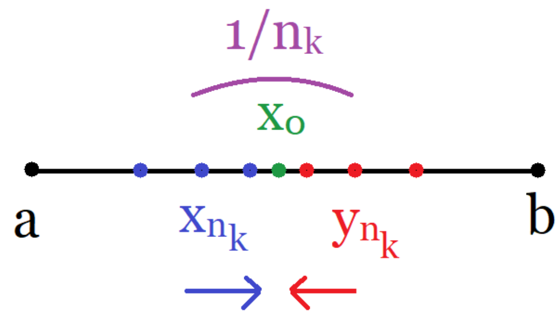
Then there is $\epsilon > 0$ such that for all $\delta > 0$ there are $x, y \in [a, b]$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$.

Then, for all $n \in \mathbb{N}$, with $\delta = \frac{1}{n}$, there are $x_n, y_n \in [a, b]$ with $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon$



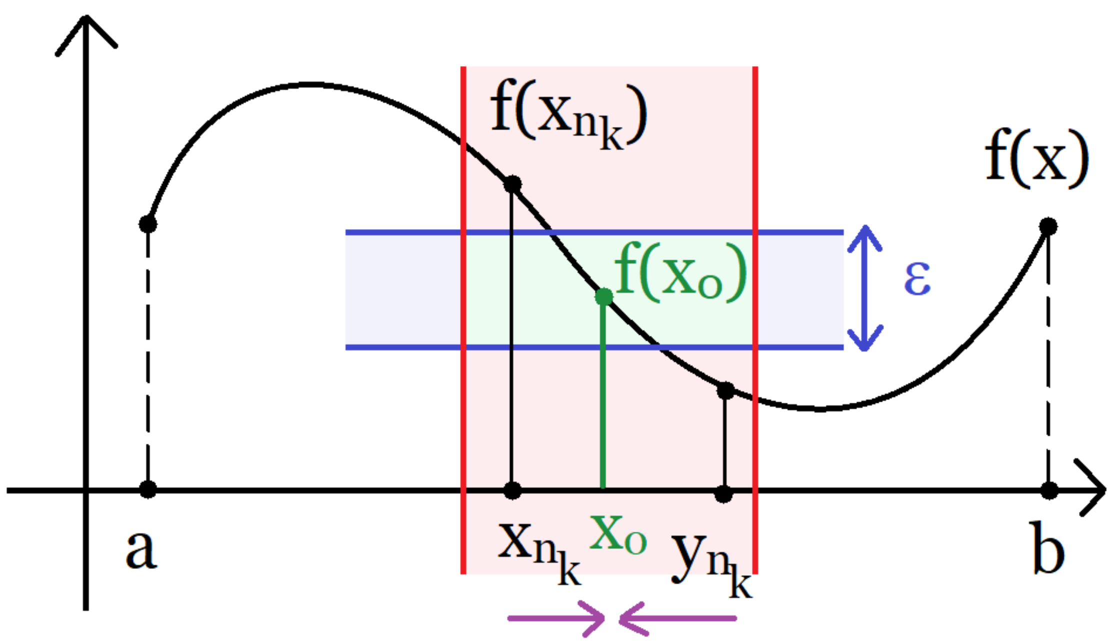
STEP 2: Let's focus on the sequence (x_n)

Since (x_n) is a sequence in $[a, b]$, (x_n) is bounded. Therefore, by Bolzano-Weierstraß, there is a subsequence (x_{n_k}) that converges to some $x_0 \in [a, b]$



By assumption, $|x_n - y_n| < \frac{1}{n}$ for all n, so $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$ for all k as well. Since $x_{n_k} \to x_0$, from this it follows that $y_{n_k} \to x_0$ as well.

STEP 3: Since $x_{n_k} \to x_0$ and f is continuous, we get $f(x_{n_k}) \to f(x_0)$. And since $y_{n_k} \to x_0$ and f is continuous, we get $f(y_{n_k}) \to f(x_0)$. Therefore letting $k \to \infty$ in $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$, we get $|f(x_0) - f(x_0)| \geq \epsilon$, so $0 \geq \epsilon > 0$, which is a contradiction



Hence f is uniformly continuous on $[a, b]$ $\square$

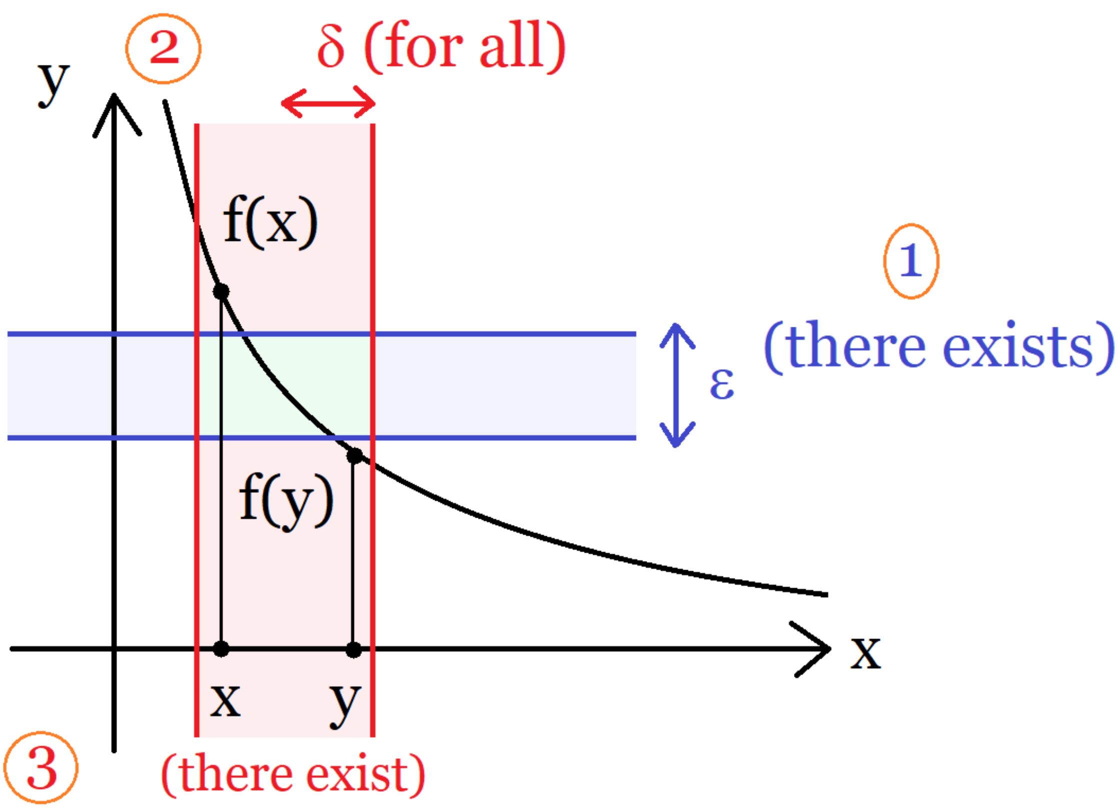
8. Optional: Not Uniformly Continuous

Video: Example 3: Not Uniformly Continuous

Just like you can only appreciate light when you see darkness, let's now discuss a function is **not** uniformly continuous:

Example 3:

Show $f(x) = \frac{1}{x}$ is **not** uniformly continuous on $(0, 1)$

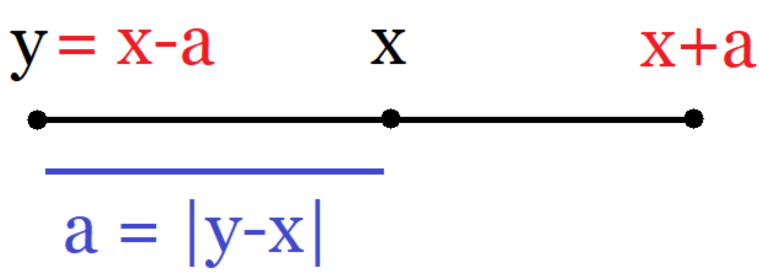


What does it mean to be *not* uniformly continuous? For this, let's recall the definition of uniform continuity:

Uniformly Continuous: For all $\epsilon > 0$ there is $\delta > 0$ such that for all $x, y \in (0,1)$, if $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon$

Not uniform continuity is just the negation of the above

Not Uniformly Continuous: There is $\epsilon > 0$ such that for all $\delta > 0$ there are $x, y \in (0,1)$ such that $|x-y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$



In other words, we need to find ϵ such that, no matter what δ we can find two evil x and y such that $|x-y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$.

STEP 1: Scratchwork

Let ϵ TBA, let $\delta > 0$ be given.

To find x, y, let's proceed as usual:

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \frac{|y-x|}{xy} \overset{?}{\geq} \epsilon$$

(Here we used $x, y > 0$ since $x, y \in (0,1)$)

STEP 2: WLOG, assume $y < x$, then $\frac{1}{y} > \frac{1}{x}$, hence

$$\frac{|y-x|}{xy} = \left(\frac{|y-x|}{x}\right)\left(\frac{1}{y}\right) \geq \left(\frac{|y-x|}{x}\right)\left(\frac{1}{x}\right) = \frac{|y-x|}{x^2} \overset{?}{\geq} \epsilon$$

STEP 3: Let $a =: |y-x| > 0$

Note that since $|y-x| < \delta$, we have $\boxed{a < \delta}$

Note that *if* we know that a and x are, then we can figure out what y is because

$$\begin{aligned} |y-x| = a \Rightarrow & y - x = \pm a \\ \Rightarrow & y = x \pm a \\ \Rightarrow & y = x - a \text{ or } y = x + a \end{aligned}$$

But since $y < x$, we get $\boxed{y = x - a}$.

STEP 4: Using $|y-x| = a$, we get

$$\frac{|y-x|}{x^2} = \frac{a}{x^2} \geq \epsilon \Rightarrow x^2 \leq \frac{a}{\epsilon} \Rightarrow x \leq \sqrt{\frac{a}{\epsilon}}$$

Let

$$x = \sqrt{\frac{a}{\epsilon}} \text{ and } y = \sqrt{\frac{a}{\epsilon}} - a$$

Upshot: *If* we know what a and ϵ are, then we know that x and y are (and we would be done)

STEP 5: Find a

For this, we need to verify that, with x and y as above, we have $x, y \in (0,1)$.

But since $\epsilon > 0$ and $a > 0$, we get $x > 0$, and moreover

$$x < 1 \Leftrightarrow \sqrt{\frac{a}{\epsilon}} < 1 \Leftrightarrow \frac{a}{\epsilon} < 1 \Leftrightarrow a < \epsilon$$

This tells us that we must choose a such that $\boxed{a < \epsilon}$ and, in that case, we have $0 < x < 1$ ✓

Now for y, first of all, since $a < \epsilon$,

$$y = \sqrt{\frac{a}{\epsilon}} - a \leq \sqrt{\frac{a}{\epsilon}} < 1 ✓$$

And

$$y > 0 \Leftrightarrow \sqrt{\frac{a}{\epsilon}} - a > 0 \Leftrightarrow a < \sqrt{\frac{a}{\epsilon}} \Leftrightarrow a^2 < \frac{a}{\epsilon} \Leftrightarrow a < \frac{1}{\epsilon}$$

If $\boxed{a < \frac{1}{\epsilon}}$, you get $y > 0$ and therefore $0 < y < 1$ ✓

Note: The *miracle* is that all of the above works no matter what ϵ is, so the proof actually works for any $\epsilon > 0$

STEP 5: Actual Proof

Let $\epsilon > 0$ be whatever you want (for example $\epsilon = 1$ works)

Let $\delta > 0$ be given

Let $a > 0$ with $a < \min\left\{\delta, \epsilon, \frac{1}{\epsilon}\right\}$ and let $x = \sqrt{\frac{\epsilon}{a}}$ and $y = \sqrt{\frac{\epsilon}{a}} - a$

Then, since $a < \epsilon$ and $a < \frac{1}{\epsilon}$, we get $x, y \in (0, 1)$.

Moreover

$$|x-y| = \left|\sqrt{\frac{\epsilon}{a}} - \left(\sqrt{\frac{\epsilon}{a}} - a\right)\right| = a < \delta$$

But

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|y-x|}{xy} \geq \frac{|y-x|}{x^2} = \frac{a}{\left(\sqrt{\frac{a}{\epsilon}}\right)^2} = \frac{\epsilon a}{a} = \epsilon \checkmark$$

Hence $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$ $\square$