LECTURE 18: LAGRANGE MULTIPLIERS (II)

Let's continue our Lagrange multiplier extravaganza!

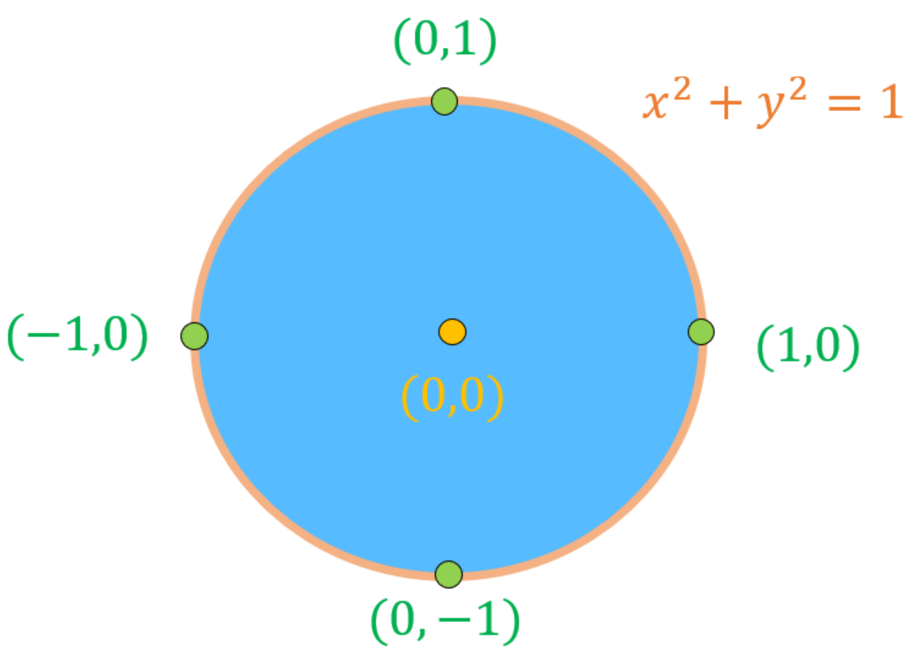
1. A full max/min problem

What if we want to find the max/min of a function in a full disk instead of just a circle?

Example 1:

Find the absolute max/min of $f(x, y)$ in the disk $x^2 + y^2 \leq 1$

$$f(x, y) = x^2 + 2y^2$$



Date: Friday, October 8, 2021.

Main Idea: Find the critical points *inside* the disk, and then use Lagrange multipliers to find the max/min on the circle

STEP 1: Critical Points

$$\begin{aligned} f_x =& 2x = 0 \Rightarrow x = 0 \\ f_y =& 4y = 0 \Rightarrow y = 0 \end{aligned}$$

So the only critical point is $\boxed{(0,0)}$ which is *inside* the disk $x^2 + y^2 \leq 1$ (if not, then ignore it)

STEP 2: Lagrange Multipliers

To find the max/Min *on* the circle $x^2 + y^2 = 1$, use Lagrange multipliers (same example as last time) to get: $\boxed{(0,1), (0,-1), (1,0), (-1,0)}$

STEP 3: Compare

$(0,0)$	$f(0,0) = 0$
$(0,-1)$	$f(0,-1) = 2$
$(0,1)$	$f(0,1) = 2$
$(1,0)$	$f(1,0) = 1$
$(-1,0)$	$f(-1,0) = 1$

Absolute Max: $f(0,1) = f(0,-1) = 2$

Absolute Min: $f(0,0) = 0$

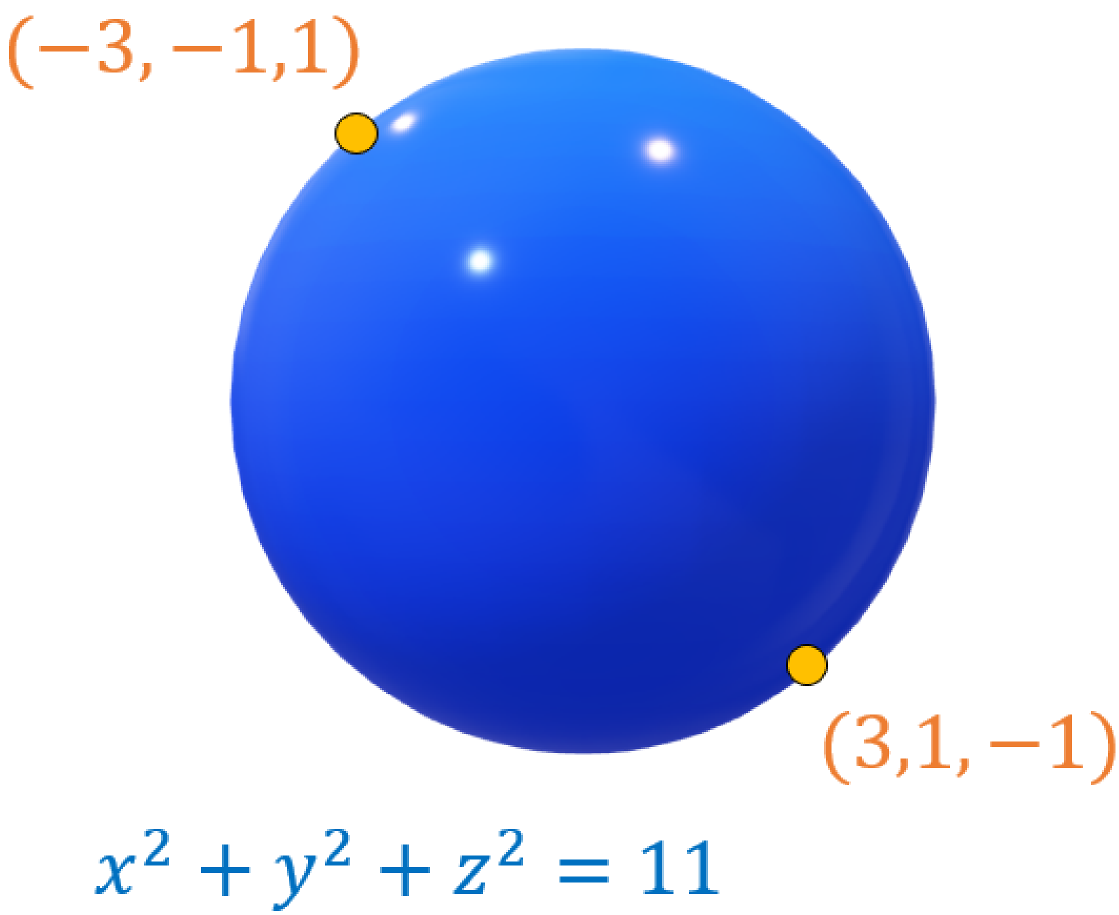
2. Three variables

The beautiful thing about Lagrange multipliers is that we can also do it in three (or more) variables

Example 2:

Find the absolute max/min of $f(x, y, z)$ on the sphere $x^2 + y^2 + z^2 = 11$

$$f(x, y, z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$



$g(x, y, z) = x^2 + y^2 + z^2 - 11$

Lagrange equation: $\nabla f = \lambda \nabla g$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ x^2 + y^2 + z^2 = 11 \end{cases} \Rightarrow \begin{cases} 2(x-3) = \lambda(2x) \\ 2(y-1) = \lambda(2y) \\ 2(z+1) = \lambda(2z) \\ x^2 + y^2 + z^2 = 11 \end{cases} \Rightarrow \begin{cases} x-3 = \lambda x \\ y-1 = \lambda y \\ z+1 = \lambda z \\ x^2 + y^2 + z^2 = 11 \end{cases}$$

Note: Last time we did this by cases. This time we explore another way of solving them. Which way to solve it depends on the situation.

From the first equation, we get:

$$x - 3 = \lambda x \Rightarrow x - \lambda x = 3 \Rightarrow x(1-\lambda) = 3 \Rightarrow x = \frac{3}{1-\lambda}$$

(Here $\lambda \neq 1$ because otherwise $x(1-\lambda) = x(0) = 0 \neq 3$, so it's ok to divide by $1 - \lambda$)

Similarly $y = \frac{1}{1-\lambda}$ and $z = \frac{-1}{1-\lambda}$

Now use the constraint:

$$\begin{aligned} x^2 + y^2 + z^2 &= 11 \\ \left(\frac{3}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{-1}{1-\lambda}\right)^2 &= 11 \\ \frac{9}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} &= 11 \\ \frac{11}{(1-\lambda)^2} &= 11 \\ (1-\lambda)^2 &= 1 \\ 1 - \lambda &= \pm 1 \end{aligned}$$

But $1-\lambda = 1 \Rightarrow \lambda = 0$ and $1-\lambda = -1 \Rightarrow \lambda = 2$

Case 1: $\lambda = 0$, then:

$$x = \frac{3}{1-\lambda} = \frac{3}{1-0} = 3, y = \frac{1}{1-\lambda} = 1, z = \frac{-1}{1-\lambda} = -1$$

Which gives the point $\boxed{(3,1,-1)}$

Case 2: $\lambda = 2$, which similarly gives the point $\boxed{(-3,-1,1)}$

Compare:

$$\begin{aligned} f(3,1,-1) &= 0 \Rightarrow \text{ Absolute Min} \\ f(-3,-1,1) &= 44 \Rightarrow \text{ Absolute Max} \end{aligned}$$

3. Another IKEA Problem

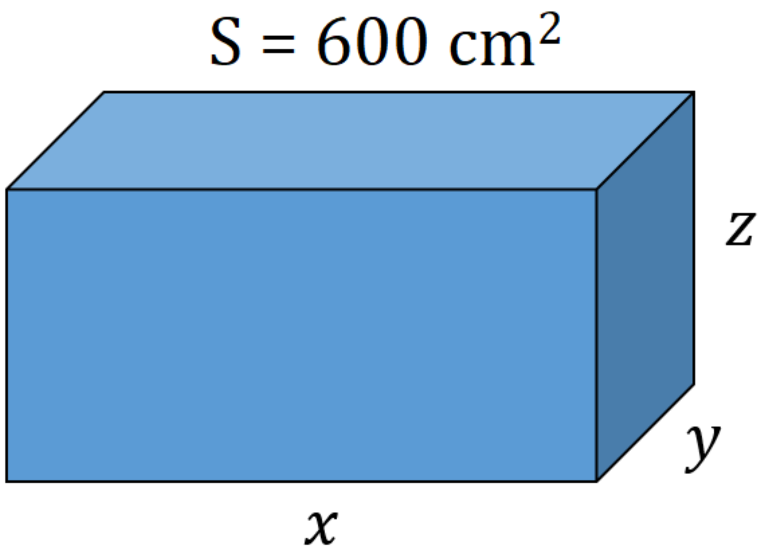
Video: Another IKEA Problem

Just like max/min, the fun with Lagrange multipliers lies in the word problems!

And with this I would like to welcome you back to IKEA! This time the box Svärsö is finally back in stock, and we would like to find its largest volume:

Example 3:

Find the largest possible volume of a box whose surface area is 600 cm^2



STEP 1: Find f and g

$$V = xyz \Rightarrow f(x, y, z) = xyz$$
$$S = 2xy + 2yz + 2xz = 600 \Rightarrow g(x, y, z) = 2xy + 2yz + 2xz - 600$$

Great because here we do **not** have to solve for z in terms of x and y

STEP 2: Lagrange equation: $\nabla f = \lambda \nabla g$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ \text{Constraint} \end{cases} \Rightarrow \begin{cases} yz = \lambda(2y + 2z) \\ xz = \lambda(2x + 2z) \\ xy = \lambda(2y + 2x) \\ 2xy + 2yz + 2xz = 600 \end{cases}$$

This looks like a nightmare, but this time, let's solve λ in terms of all the variables

$$yz = \lambda(2y + 2z) \Rightarrow \lambda = \frac{yz}{2y + 2z} \tag{1}$$

And similarly, from the second equation

$$\lambda = \frac{xz}{2x+2z} \tag{2}$$

And from the third equation

$$\lambda = \frac{xy}{2x+2y} \tag{3}$$

Trick: Since all quantities are equal to λ, they are equal to each other!

$$\begin{aligned}
(1) &= (2) \\
\frac{y\cancel{z}}{2y+2z} &= \frac{x\cancel{z}}{2x+2z} \\
y(2x+2z) &= x(2y+2z) \\
\cancel{2xy} + 2yz &= \cancel{2xy} + 2xz \\
2yz &= 2xz \\
y &= x
\end{aligned}$$

Similarly (2) = (3) gives $y = z$

Conclusion: $x = y = z$, so the optimal box is a cube!

STEP 3: Constraint:

$$\begin{aligned}
2xy + 2yz + 2xz &= 600 \\
2x^2 + 2x^2 + 2x^2 &= 600 \qquad \text{Since } x = y = z \\
6x^2 &= 600 \\
x^2 &= 100 \\
x &= 10 \qquad (x > 0)
\end{aligned}$$

Hence $x = 10, y = 10, z = 10$

STEP 4: Therefore the absolute max is:

$$f(10,10,10) = (10)(10)(10) = 1000 \text{ cm}^{3}$$

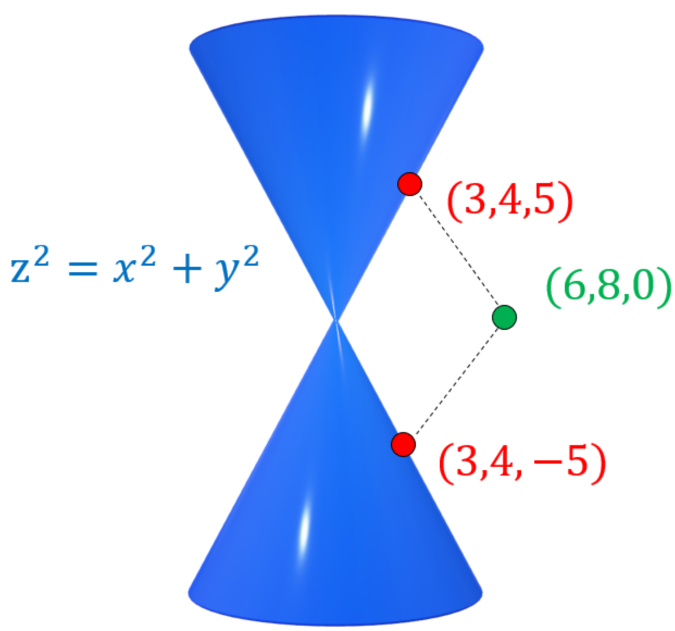
Extra Practice:

Use Lagrange multipliers to find the **smallest** possible surface area of a box whose volume is 1000 cm^3

4. Distance Problem

Example 4:

Find the point(s) on the cone $z^2 = x^2 + y^2$ that are closest to $(6,8,0)$



STEP 1: Find f and g

$$f(x,y,z) = \text{Distance}^{\,2} = (x-6)^2 + (y-8)^2 + z^2$$
$$g(x,y,z) = \text{Cone } = x^2 + y^2 - z^2$$

STEP 2: Lagrange Equation: $\nabla f = \lambda \nabla g$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ x^2 + y^2 = z^2 \end{cases} \Rightarrow \begin{cases} 2(x-6) = \lambda(2x) \\ 2(y-8) = \lambda(2y) \\ 2z = \lambda(-2z) \\ x^2 + y^2 = z^2 \end{cases} \Rightarrow \begin{cases} x - 6 = \lambda x \\ y - 8 = \lambda y \\ z = -\lambda z \\ x^2 + y^2 = z^2 \end{cases}$$

Look at the easiest equation, which is:

$$z = -\lambda z \Rightarrow z(1+\lambda) = 0 \Rightarrow z = 0 \text{ or } \lambda = -1$$

Here we have to do it by cases:

Case 1: $z = 0$

But then $x^2 + y^2 = z^2 \Rightarrow x^2 + y^2 = 0 \Rightarrow x = 0$ and $y = 0$, but then the first equation becomes:

$$(x-6) = \lambda x \Rightarrow (0-6) = \lambda 0 \Rightarrow -6 = 0$$

Which is impossible ✗

Case 2: $\lambda = -1$

But then the first equation becomes:

$$\begin{aligned} x - 6 =& \lambda x \\ x - 6 =& - x \\ 2x =& 6 \\ x =& 3 \end{aligned}$$

And the second equation becomes:

$$\begin{aligned} y - 8 =& \lambda y \\ y - 8 =& - y \\ 2y =& 8 \\ y =& 4 \end{aligned}$$

And then using $x^2 + y^2 = z^2$ gives $3^2 + 4^2 = z^2$ so $z^2 = 25$ so $z = \pm 5$, which gives the points $\boxed{(3, 4, 5), (3, 4, -5)}$

STEP 3: Compare:

$$\begin{aligned} f(3, 4, 5) =& (3 - 6)^2 + (4 - 8)^2 + 5^2 = 3^2 + 4^2 + 5^2 = 50 \\ f(3, 4, -5) =& (3 - 6)^2 + (4 - 8)^2 + (-5)^2 = 50 \end{aligned}$$

The points on the cone $z^2 = x^2 + y^2$ closest to $(6, 8, 0)$ are $(3, 4, 5)$ and $(3, 4, -5)$ (and the smallest distance would be $\sqrt{50}$)