LECTURE 18: THE LEBESGUE MEASURE

Welcome to the world of measure theory, where we'll learn the correct way of finding the size of a set[1]

1. Rectangles

The idea is to build more complicated sets from simple ones, the most basic unit being a rectangle:

Definition: If $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ is a rectangle in $\mathbb{R}^d$ then

$$|R| =: (b_1 - a_1) \cdots (b_d - a_d)$$

Definition: A union of rectangles is **almost disjoint** if the interiors of the rectangles are disjoint

This just means they're disjoint but their sides may overlap (see picture in lecture)

Fact: If R is the almost disjoint union of finitely many rectangles, $R = \bigcup_{k=1}^{N} R_k$ then

$$|R| = \sum_{k=1}^{N} |R_k|$$

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[1]We will follow the presentation of Stein and Shakarchi's *Real Analysis* book (Book 3), which gives a more hands-on approach to measure theory compared to the one in Rudin

Proof: Consider the grid formed by extending the sides of the rectangles $R_1, \dots, R_N$ (see picture in lecture)

We thereby obtain finitely many rectangles $\widetilde{R_1}, \cdots, \widetilde{R_M}$ (for some M) and a partition $J_1, \dots, J_N$ of $\{1, \dots, M\}$ such that the following unions are almost disjoint

$$R = \bigcup_{j=1}^{M} \tilde{R}_j \text{ and } R_k = \bigcup_{j \in J_k} \tilde{R}_j$$

(in the picture in lecture, we have $R_1 = \tilde{R}_1 \cup \tilde{R}_2$ and $J_1 = \{1, 2\}$)

$$\text{Hence } |R| = \sum_{j=1}^{M} \left|\tilde{R}_j\right| = \sum_{k=1}^{N} \left(\sum_{j \in J_k} \left|\tilde{R}_j\right| \right) = \sum_{k=1}^{N} |R_k|$$

The first equality follows[2] since the $\tilde{R}_j$ partition the rectangle R, the follows since the J_k partition $\{1, \cdots, M\}$, and the last one follows since the $\tilde{R}_j$ partition R_k □

Corollary: If R and $R_1, \cdots R_N$ are rectangles and $R \subseteq \bigcup_{k=1}^{N} R_k$ then

$$|R| \leq \sum_{k=1}^{N} |R_k|$$

Proof: Same proof as the above, but notice that the sets corresponding to the J_k are not disjoint any more.

2. The Exterior Measure

[2] If you're not convinced, use the definition of $|R|$ and $|R_j|$ in terms of sums of products of numbers

For more complicated sets, the idea is to cover the set with closed cubes, similar to what you did when you discussed compactness.

Definition: If E is *any* subset of $\mathbb{R}^d$ then the **exterior measure** of E is

$$m_\star(E) =: \inf \sum_{j=1}^{\infty} |Q_j|$$

Where the infimum is taken over all countable coverings $E \subseteq \bigcup_{j=1}^{\infty} Q_j$ with closed cubes.

Note: With this definition, $m_\star(E)$ always exists, no matter how crazy E is, although it could be ∞.

Note: We're using cubes because they're simple enough to deal with. But it would be totally ok in theory to use rectangles, and even balls.

Example/Fact: If Q is a closed cube, then $m_\star(Q) = |Q|$

Since Q covers itself we have $m_\star(Q) \leq |Q|$

For the reverse inequality, let $Q \subseteq \bigcup_{j=1}^{\infty} Q_j$ be an arbitrary covering by cubes, and show $|Q| \leq \sum_{j=1}^{\infty} |Q_j|$, because then take the inf on the right-hand-side.

Let $\epsilon > 0$ be given, then for each j find a slightly bigger open cube $S_j \supseteq Q_j$ such that $|S_j| \leq (1+\epsilon)\,|Q_j|$ (so S_j is bigger, but not too big)

Notice $\bigcup_{j=1}^{\infty} S_j$ covers Q (compact) so by compactness, we can find a finite sub-covering, which we relabel as $Q \subseteq \bigcup_{j=1}^{N} S_j$.

Then by the above rectangle fact we have

$$|Q| \leq \sum_{j=1}^{N} |S_j| \overset{\text{DEF}}{\leq} \sum_{j=1}^{N} (1+\epsilon)\,|Q_j| \leq (1+\epsilon)\sum_{j=1}^{N} |Q_j| \leq (1+\epsilon)\sum_{j=1}^{\infty} |Q_j|$$

Since $\epsilon > 0$ is arbitrary we find $|Q| \leq \sum_{j=1}^{\infty} |Q_j|$ as desired □

Note: The same result holds if Q is an open cube (since Q is covered by $\overline{Q}$) and if R is a rectangle (in that case you form a grid formed by cubes). Finally $m_\star(\mathbb{R}^d) = \infty$ since $\mathbb{R}^d$ includes arbitrarily large cubes.

3. Properties of the Outer Measure

First of all, from the definition of $m_\star$ as an inf, we have

Property 0:

For all $\epsilon > 0$ there is a covering $E \subseteq \bigcup_{j=1}^{\infty} Q_j$ such that $\sum_{j=1}^{\infty} |Q_j| \leq m_\star(E) + \epsilon$

Note: We will use this many many times. We're basically saying $\sum_{j=1}^{\infty} |Q_j|$ is not *much* bigger than $m_\star(E)$

Property 1: (Monotonicity) If $E_1 \subseteq E_2$ then $m_\star(E_1) \leq m_\star(E_2)$

This is because a covering of E_2 by cubes is also a covering of E_1, so there are more sets to consider in the inf of E_1

In particular, every bounded set has finite outer measure, since you that set is included in a large cube.

Property 2: (Countable sub-additivity)

$$\text{If } E = \bigcup_{j=1}^{\infty} E_j \text{ then } m_\star(E) \leq \sum_{j=1}^{\infty} m_\star(E_j)$$

This is kind of like a triangle inequality for the outer measure

Proof: This method of proof is called a $\frac{\epsilon}{2^n}$ argument and appears over and over again in measure theory.

WLOG, assume $m_\star(E_j) < \infty$ for each j and let $\epsilon > 0$ be given

Then there is a covering $E_j \subseteq \bigcup_{k=1}^{\infty} Q_{k,j}$ by closed cubes such that

$$\sum_{k=1}^{\infty} |Q_{k,j}| \leq m_\star(E_j) + \frac{\epsilon}{2^j}$$

But then $\bigcup_{j,k=1}^{\infty} Q_{k,j}$ is a covering of E by closed cubes, and hence

$$m_\star(E) \overset{\text{inf}}{\leq} \sum_{j,k=1}^{\infty} |Q_{k,j}| = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |Q_{k,j}| \leq \sum_{j=1}^{\infty} \left(m_\star(E_j) + \frac{\epsilon}{2^j} \right) \overset{\text{Geom.}}{=} \left(\sum_{j=1}^{\infty} m_\star(E_j) \right) + \epsilon$$

Since $\epsilon > 0$ is arbitrary, we are done $\square$

Property 3: (Open sets)

$$m_\star(E) = \inf m_\star(O)$$

Where the inf is taken over all open sets O containing E.

So not just all cube-coverings, but literally all open set coverings!

Proof: Since $E \subseteq O$ we have $m_\star(E) \leq m_\star(O)$ and taking the inf over O we have $m_\star(E) \leq \inf m_\star(O)$ ✓

For the reverse inequality, let $\epsilon > 0$ be given and choose a covering of closed cubes Q_j such that $\sum_{j=1}^{\infty} |Q_j| \leq m_\star(E) + \frac{\epsilon}{2}$

Let $S_j \supseteq Q_j$ be an open cube containing Q_j such that $|S_j| \leq |Q_j| + \frac{\epsilon}{2^{j+1}}$ then $O =: \bigcup_{j=1}^{\infty} S_j$ is open and so by countable sub-additivity

$$\begin{aligned} m_\star(O) \leq \sum_{j=1}^{\infty} m_\star(S_j) = \sum_{j=1}^{\infty} |S_j| \overset{\text{DEF}}{\leq} \sum_{j=1}^{\infty} |Q_j| + \frac{\epsilon}{2^{j+1}} \overset{\text{GEOM}}{=} \sum_{j=1}^{\infty} |Q_j| + \frac{\epsilon}{2} \\ \leq & m_\star(E) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = m_\star(E) + \epsilon ✓ \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we have $\inf m_\star(O) \leq m_\star(E)$ □

Warning: If E_1 and E_2 are disjoint, then in general we do **not** have

$$m_\star(E_1 \cup E_2) \neq m_\star(E_1) + m_\star(E_2)$$

It *is* true provides E_1 and E_2 are a positive distance from each other

Definition: $d(E_1, E_2) = \inf \{|x - y|, x \in E_1, y \in E_2\}$

Property 4: If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$ then

$$m_\star(E_1 \cup E_2) = m_\star(E_1) + m_\star(E_2)$$

Proof: We already know $m_\star(E) \leq m_\star(E_1) + m_\star(E_2)$ by countable sub-additivity

For the reverse inequality, choose $\delta > 0$ be so that $d(E_1, E_2) > \delta > 0$.

Let $\epsilon > 0$ be given and choose a covering Q_j so that $\sum_{j=1}^{\infty} |Q_j| \leq m_\star(E) + \epsilon$

After sub-dividing the cubes if necessary, assume that each Q_j has diagonal less than δ.

Since E_1 and E_2 are a distance at least δ apart, each Q_j can intersect at most one of the two sets E_1 or E_2. Let J_1 and J_2 be the set of those indices, so

$$E_1 \subseteq \bigcup_{j \in J_1} Q_j \text{ and } E_2 \subseteq \bigcup_{j \in J_2} Q_j$$

$$m_\star(E_1) + m_\star(E_2) \leq \sum_{j \in J_1} |Q_j| + \sum_{j \in J_2} |Q_j| = \sum_{j=1}^{\infty} |Q_j| \leq m_\star(E) + \epsilon$$

We are done since $\epsilon > 0$ was arbitrary

Property 5: If E is the countable union of almost disjoint cubes, $E = \bigcup_{j=1}^{\infty} Q_j$ then

$$m_\star(E) = \sum_{j=1}^{\infty} |Q_j|$$

Proof: We only need to show $\geq$ since $\leq$ follows from countable subadditivity

Let $\epsilon > 0$ be given and let $S_j \subseteq Q_j$ be a closed cube strictly contained in Q_j such that $|Q_j| \leq |S_j| + \frac{\epsilon}{2^j}$

Then for every N, the cubes $S_1, S_2, \cdots S_N$ are disjoint, hence of finite distance from each other, and therefore by the above, we get

$$m_\star\left(\bigcup_{j=1}^{N} S_j\right) = \sum_{j=1}^{N} |S_j| \overset{\text{DEF}}{\geq} \sum_{j=1}^{N}\left(|Q_j| - \frac{\epsilon}{2^j}\right) \overset{\text{GEOM}}{\geq} \sum_{j=1}^{N} |Q_j| - \epsilon$$

Since $\bigcup_{j=1}^{N} S_j \subseteq E$ we get that for every N, $m_\star(E)$ is even bigger, so

$$m_\star(E) \geq \sum_{j=1}^{N} |Q_j| - \epsilon$$

Letting $N \to \infty$ we then get $m_\star(E) \geq \sum_{j=1}^{\infty} |Q_j| - \epsilon$ and we're done because ϵ is arbitrary □

Note: It can be shown that any open set is the countable union of almost disjoint cubes[3], so this would actually give us a procedure of calculating the outer measure of any open set.

4. The Lebesgue Measure

Even though the outer measure $m_\star$ works for all sets, in practice we would like to exclude some pathological sets that are "not measurable."

Definition: A subset E in $\mathbb{R}^d$ is **Lebesgue mesurable** if for every $\epsilon > 0$ there is an open set O with $E \subseteq O$ and

$$m_\star(O - E) \leq \epsilon$$

In other words, E can be well-approximated with open sets

Definition: If E is measurable, then the **Lebesgue Measure** $m(E)$

[3] See Stein and Shakarchi Theorem 1.4 in Chapter 1

$$\text{is simply } m(E) =: m_\star(E)$$

So the Lebesgue measure is just the outer measure, but restricted to a special class of sets

Property 1: Every open set in $\mathbb{R}^d$ is measurable

Why? Just let $O = E$