OPTIONAL: PROOFS OF FACTS

Here are the proofs of the facts stated in lecture.

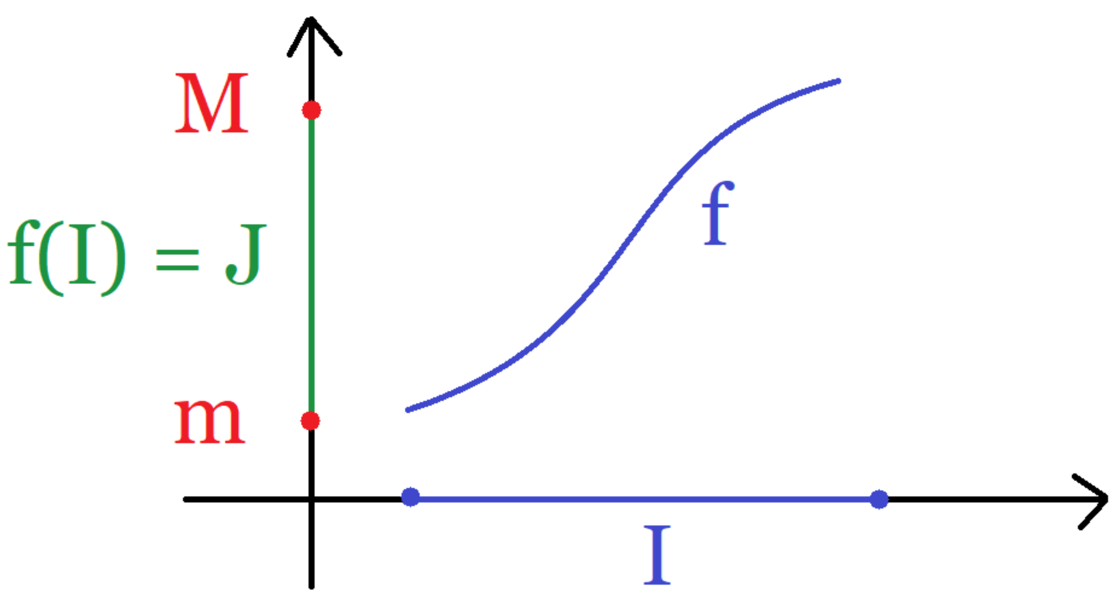
1. Image of an interval

Video: Image of an interval

Theorem 1:

If f is continuous, then $f(I)$ is an interval (or a single point)

Proof: Let $J =: f(I)$ and let $m =: \inf(J)$ and $M =: \sup(J)$

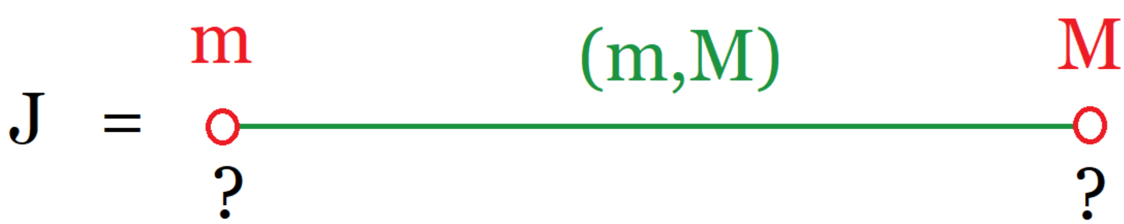


Date: Thursday, October 28, 2021.

Case 1: $m = M$, then $J = \{m\}$ is a single point ✓

Case 2: $m < M$.

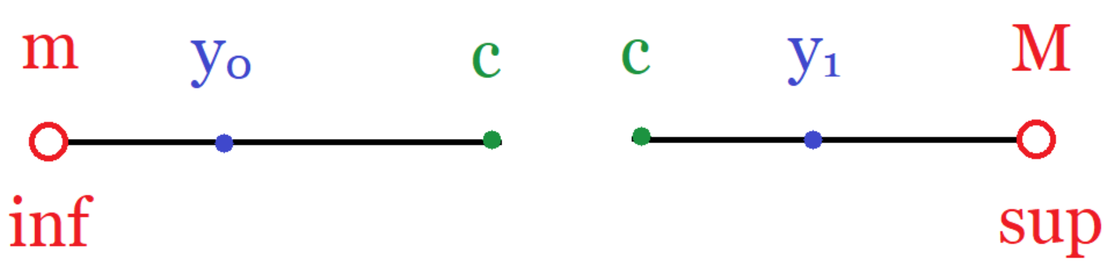
Claim: J contains the interval (m, M)



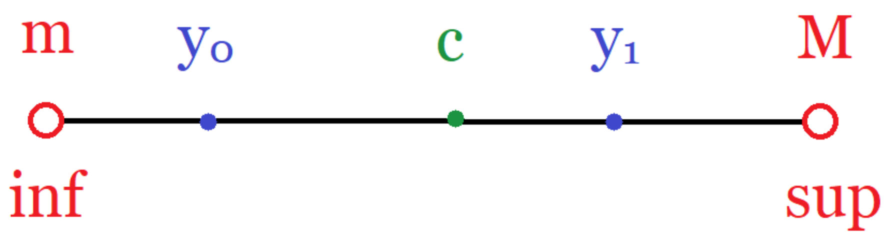
Then we would be done because we would then have either $J = (m, M)$ or $J = [m, M)$ or $J = (m, M]$ or $J = [m, M]$, depending on whether or not $m = \inf(J)$ and $M = \sup(J)$ are in J or not (here the endpoints may be infinite).

Proof of Claim: Let $c \in (m, M)$, and show $c \in J$.

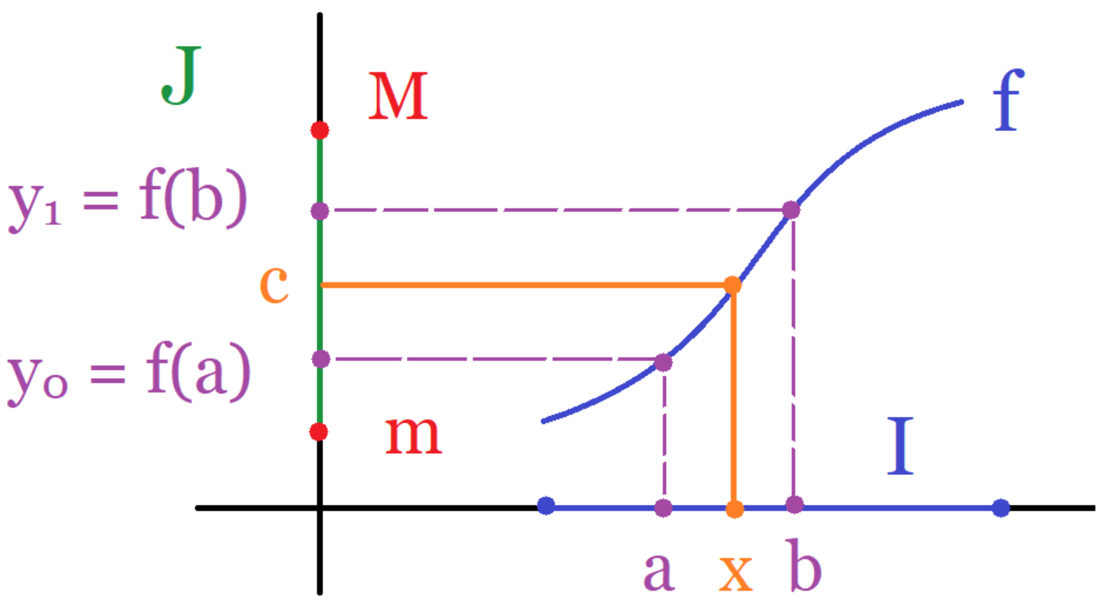
By assumption $m < c < M$. Since $c > m = \inf(J)$, by definition of inf, there is $y_0 \in J$ such that $y_0 < c$, and since $c < M = \sup(J)$, there is $y_1 \in J$ such that $c < y_1$.



Therefore we get $y_0 < c < y_1$.



Since $y_0 \in J = f(I)$, by definition of $f(I)$, there is $a \in I$ such that $y_0 = f(a)$. Similarly there is $b \in I$ such that $y_1 = f(b)$.



Since f is continuous and c is between $f(a)$ and $f(b)$, by the Intermediate Value Theorem, there is x between $a \in I$ and $b \in I$ (so $x \in I$ since I is an interval) such that $f(x) = c$, but this means that $c \in f(I) = J$ $\square$

2. Continuous Functions are Monotonic

Video: Continuity and Monotonicity

Theorem 2:

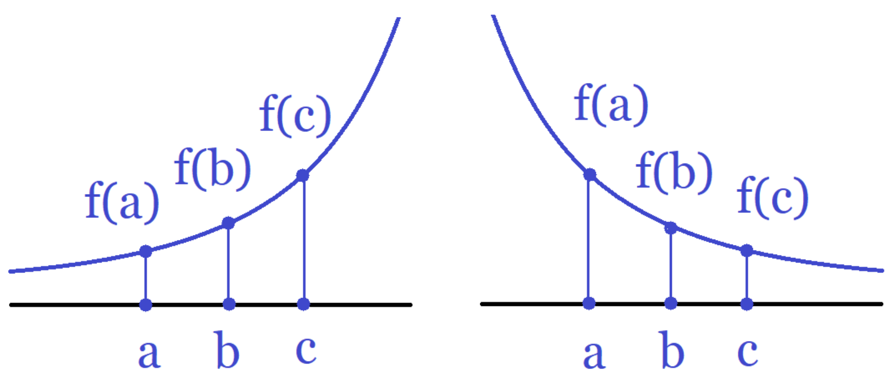
If $f : I \to \mathbb{R}$ is one-to-one and continuous, then f must be monotonic

Proof: Suppose f is continuous and one-to-one.

STEP 1:

Claim: For all $a < b < c$

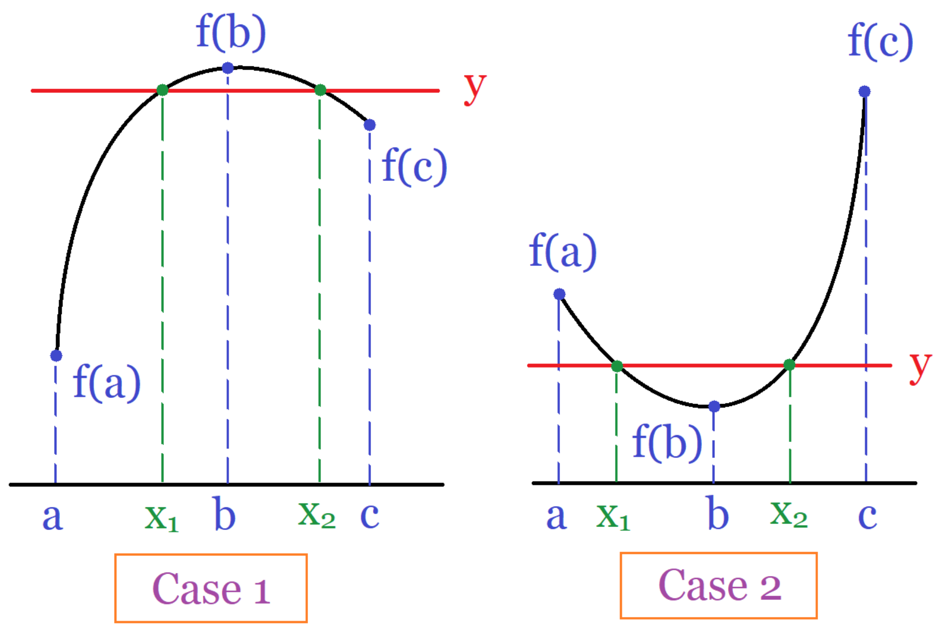
Either $f(a) < f(b) < f(c)$ or $f(a) > f(b) > f(c)$



Suppose not, then for some $a < b < c$ we have

(1) $f(b) \geq f(a)$ and $f(b) \geq f(c)$, or

(2) $f(b) \leq f(a)$ and $f(b) \leq f(c)$



(The picture illustrates the cases where $f(c) > f(a)$, but the cases where $f(c) < f(a)$ are similar)

WLOG, assume (1), that is $f(b) \geq f(a)$ and $f(b) \geq f(c)$ (the other case is similar)

Since f is one-to-one, we have $f(b) \neq f(a)$ and $f(b) \neq f(c)$, hence (1) becomes $f(b) > f(a)$ and $f(b) > f(c)$.

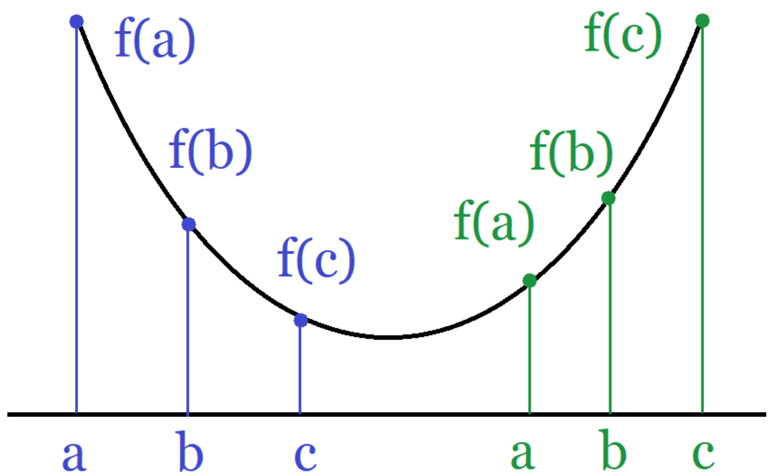
Let y be a number that is both (strictly) between $f(a)$ and $f(b)$ and between $f(b)$ and $f(c)$.

Since f is continuous, by the IVT on (a, b), there must be x_1 in (a, b) such that $f(x_1) = y$. And by the IVT on (b, c) there must be x_2 in (b, c) with $f(x_2) = y$

But then $f(x_1) = f(x_2) = y$ whereas $x_1 \neq x_2$, which contradicts f being one-to-one $\Rightarrow\Leftarrow$ ✓

STEP 2: Therefore, for all $a < b < c$, either $f(a) < f(b) < f(c)$ or $f(a) > f(b) > f(c)$.

Problem: In theory have a function f, we have $f(a) < f(b) < f(c)$ for *some* $a < b < c$, and $f(a) > f(b) > f(c)$ for *other* $a < b < c$, which is not monotonic, as in the following picture:

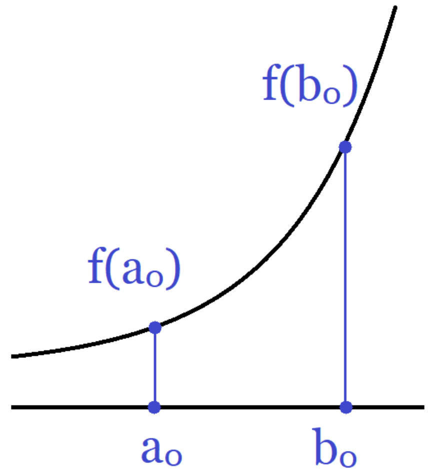


Essentially, we need to rule out one of the two possibilities.

Fix $a_0 < b_0$ in I (Think of a_0 and b_0 as *helper* numbers because they help us determine if f is increasing or decreasing. In the $\sin(x)$ example above, $a_0 = 0$ and $b_0 = \frac{\pi}{2}$)

Since f is one-to-one, we have $f(a_0) \neq f(b_0)$, hence either $f(a_0) < f(b_0)$ or $f(a_0) > f(b_0)$.

Assume WLOG $f(a_0) < f(b_0)$ (the other case is similar but would give you f decreasing)

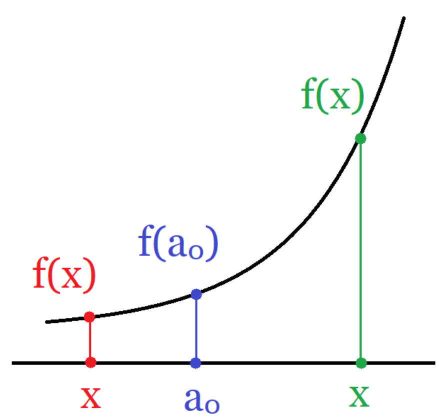


Goal: Show f is increasing.

STEP 3: Let $x \in I$

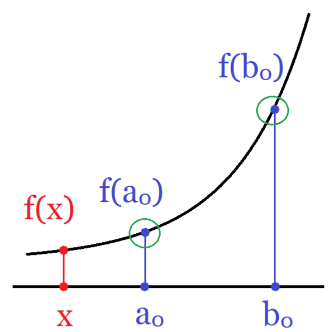
Claim:

$$x < a_0 \Rightarrow f(x) < f(a_0)$$
$$x > a_0 \Rightarrow f(x) > f(a_0)$$



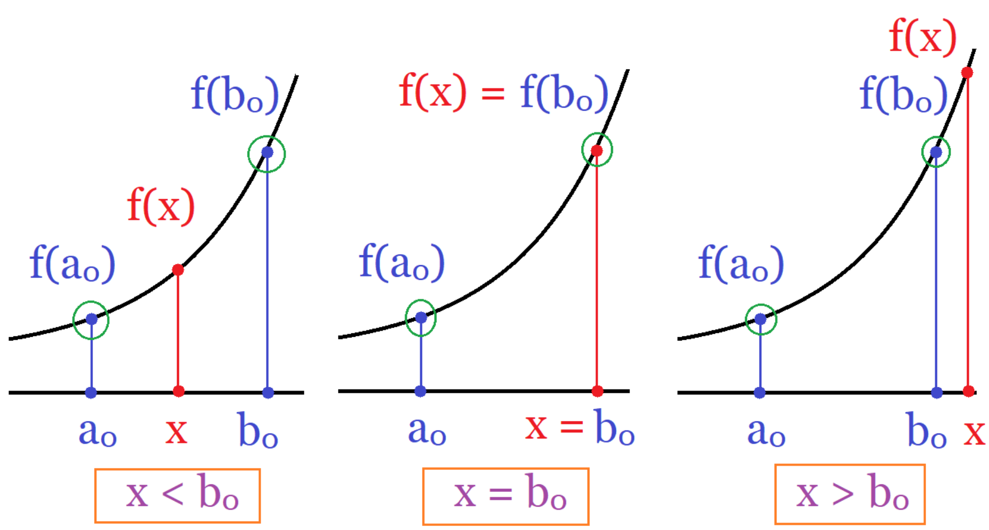
(This is *not quite* the same as f being increasing increasing, since a_0 is fixed here)

Case 1: $x < a_0$



Then, since $x < a_0 < b_0$ and $f(a_0) < f(b_0)$, by STEP 1, we must have $f(x) < f(a_0) < f(b_0)$ so $f(x) < f(a_0)$ ✓

Case 2: $x > a_0$



Case 2a: If $a_0 < x < b_0$, then, similar to Case 1, we get $f(a_0) < f(x) < f(b_0)$ so $f(x) > f(a_0)$ ✓

Case 2b: If $x = b_0$, then we get $f(x) = f(b_0) > f(a_0)$ ✓

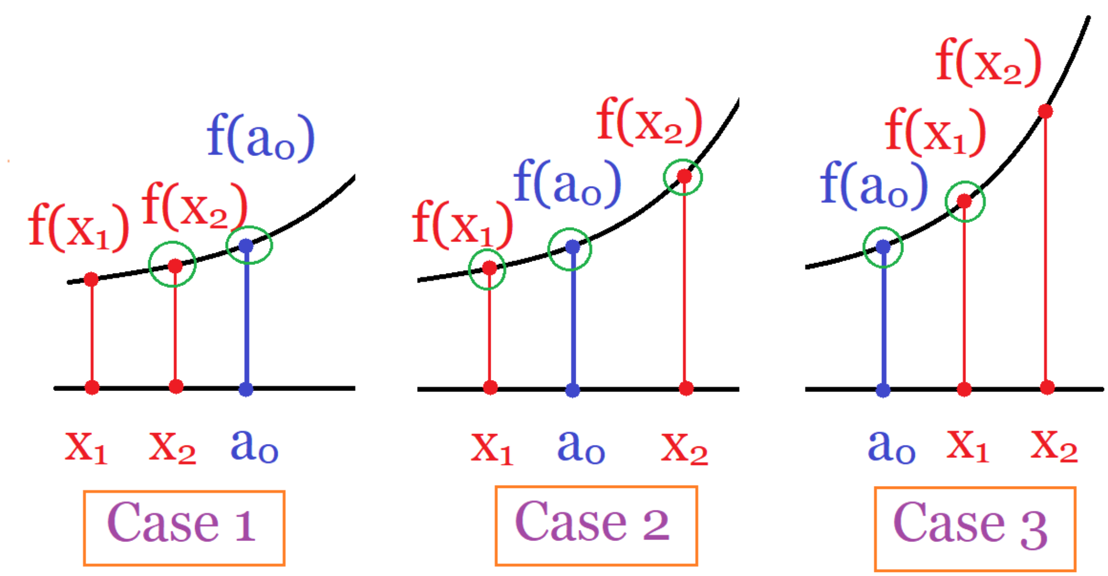
Case 2c: If $x > b_0$, then since $a_0 < b_0 < x$ and therefore $f(a_0) < f(b_0) < f(x)$, so $f(x) > f(a_0)$ ✓

Therefore we get $f(x) > f(a_0)$ □

STEP 4:

Claim: f is increasing

Suppose $x_1 < x_2$ and show $f(x_1) < f(x_2)$



Case 1: $x_1 < x_2 < a_0$

Since $x_2 < a_0$ then STEP 3 implies $f(x_2) < f(a_0)$, and therefore from STEP 1, have $f(x_1) < f(x_2) < f(a_0)$, and hence $f(x_1) < f(x_2)$ ✓

Case 2: $x_1 \leq a_0 \leq x_2$

Since $x_1 \leq a_0$, we get[1] $f(x_1) \leq f(a_0)$, and since $x_2 \geq a_0$ we get $f(x_2) \geq f(a_0)$, and therefore $f(x_1) \leq f(a_0) \leq f(x_2)$, hence $f(x_1) \leq f(x_2)$. Moreover, since $x_1 \neq x_2$ and f is one-to-one we have $f(x_1) \neq f(x_2)$. Hence $f(x_1) < f(x_2)$ ✓

Case 3: $a_0 < x_1 < x_2$.

Since $a_0 < x_1$ we get $f(a_0) < f(x_1)$ from STEP 3, and therefore, since $a_0 < x_1 < x_2$, we get $f(a_0) < f(x_1) < f(x_2)$ and hence $f(x_1) < f(x_2)$ ✓

In either case, we get that f is increasing $\square$

3. f^{-1} IS CONTINUOUS

Video: f^{-1} is continuous

Theorem 3:

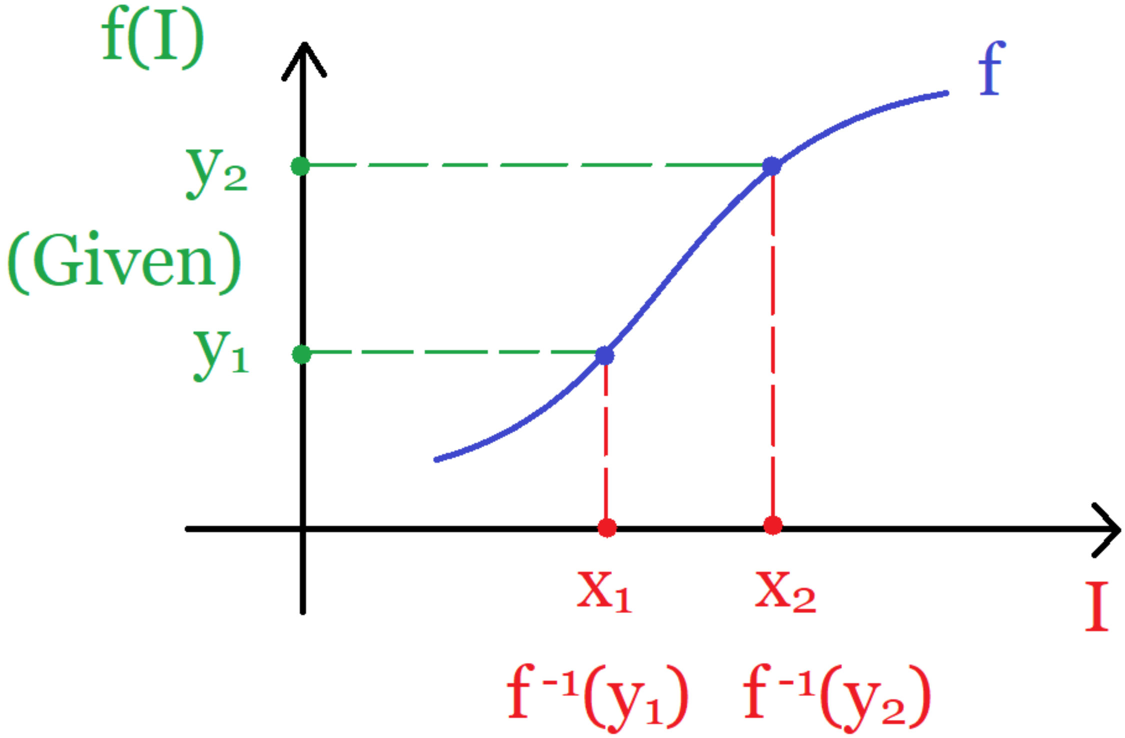
If $f : I \to f(I)$ is one-to-one and continuous, then $f^{-1} : f(I) \to I$ is continuous as well

[1] If $x_1 < a_0$ then $f(x_1) < f(a_0)$ and if $x_1 = a_0$ we get $f(x_1) = f(a_0)$ in which case the inequality holds

Lemma:

If $f : I \to f(I)$ is increasing then $f^{-1} : f(I) \to I$ is also increasing

Proof of Lemma: Suppose $y_1, y_2 \in f(I)$ are such that $y_1 < y_2$. We need to show $f^{-1}(y_1) < f^{-1}(y_2)$



By definition of $f(I)$, there is $x_1 \in I$ with $y_1 = f(x_1)$ and there is $x_2 \in I$ with $y_2 = f(x_2)$. In particular $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$

Now if $x_1 \geq x_2$, since f is increasing, we would have $f(x_1) \geq f(x_2)$ and therefore, by definition $y_1 \geq y_2 \Rightarrow\Leftarrow$

Therefore we must have $x_1 < x_2$, that is $f^{-1}(y_1) < f^{-1}(y_2)$.

We have shown that $y_1 < y_2 \Rightarrow f^{-1}(y_1) < f^{-1}(y_2)$, and therefore f^{-1} is increasing $\square$

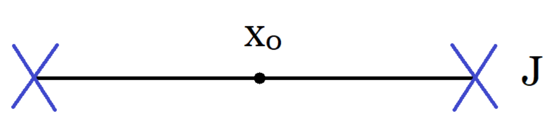
Proof of Theorem: Since f is continuous, from the previous section, f is either increasing or decreasing, so WLOG, assume f is increasing.

To simplify notation, let $J = f(I)$ and $g = f^{-1}$

Goal: Prove that for all $x_0 \in J$, g is continuous at x_0

STEP 1: First of all, since I is an interval and f is continuous, then $J = f(I)$ is also interval (from last time)

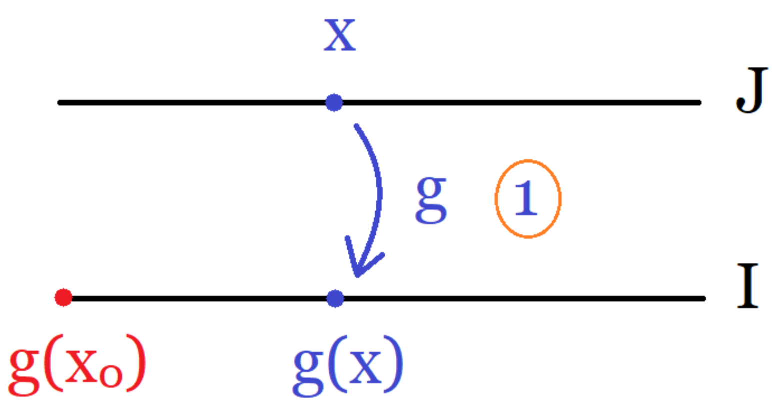
For simplicity, assume that x_0 is not an endpoint of J (for example if $J = [2, 3]$, assume x_0 is neither 2 or 3). The general case is similar.



Claim: $g(x_0)$ is not an endpoint of I

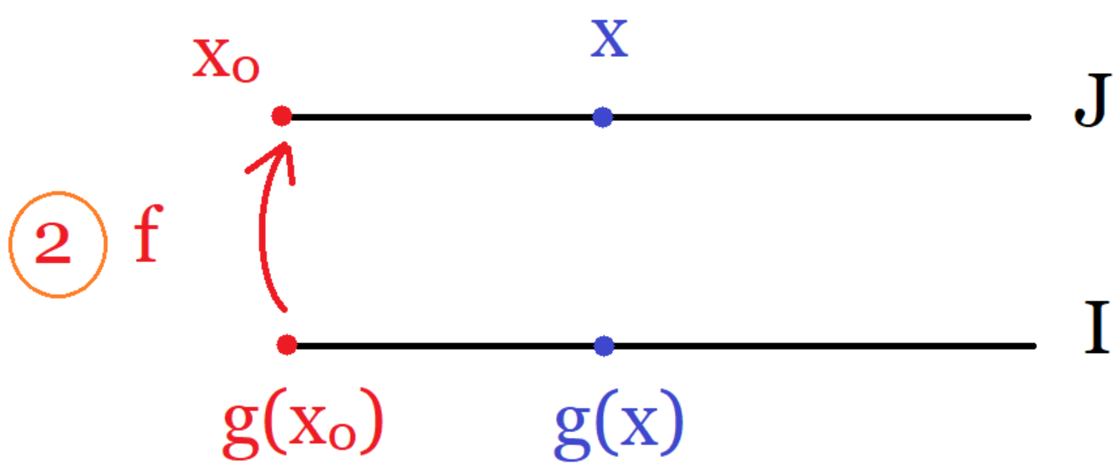
Proof of Claim: Suppose not, and assume for example that $g(x_0)$ is the left endpoint of I.

Let $x \in J$ be arbitrary. Then $g(x) \in I$, and, since $g(x_0)$ is the left endpoint of I, we must have $g(x_0) \leq g(x)$



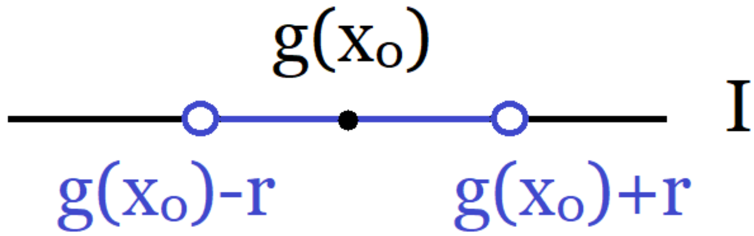
But then, since f is increasing and $g = f^{-1}$, we get

$$f\left(g(x_0)\right) \le f\left(g(x)\right) \Rightarrow x_0 \le x$$



But then this means that x_0 is the left endpoint of J, which is a contradiction $\Rightarrow\Leftarrow$ $\square$

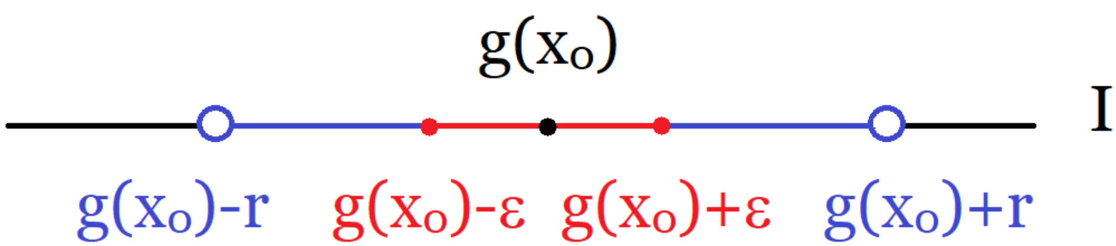
Since $g(x_0)$ is not an endpoint of I, it must be in the interior of I, and so there exists $r > 0$ such that $(g(x_0) - r, g(x_0) + r) \subseteq I$



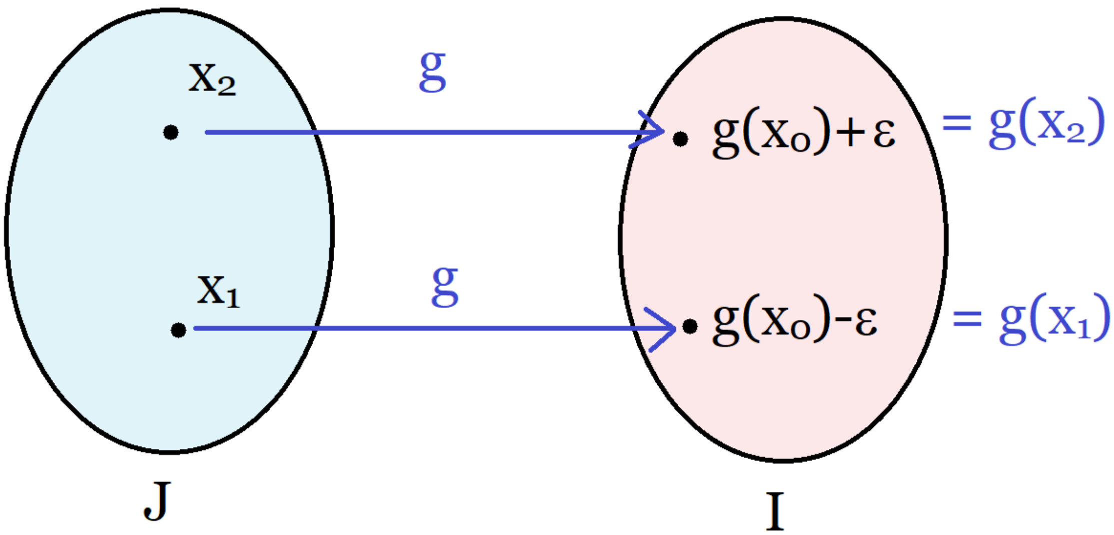
STEP 2: Let $\epsilon > 0$ be given, and assume ϵ so small that $\epsilon < r$

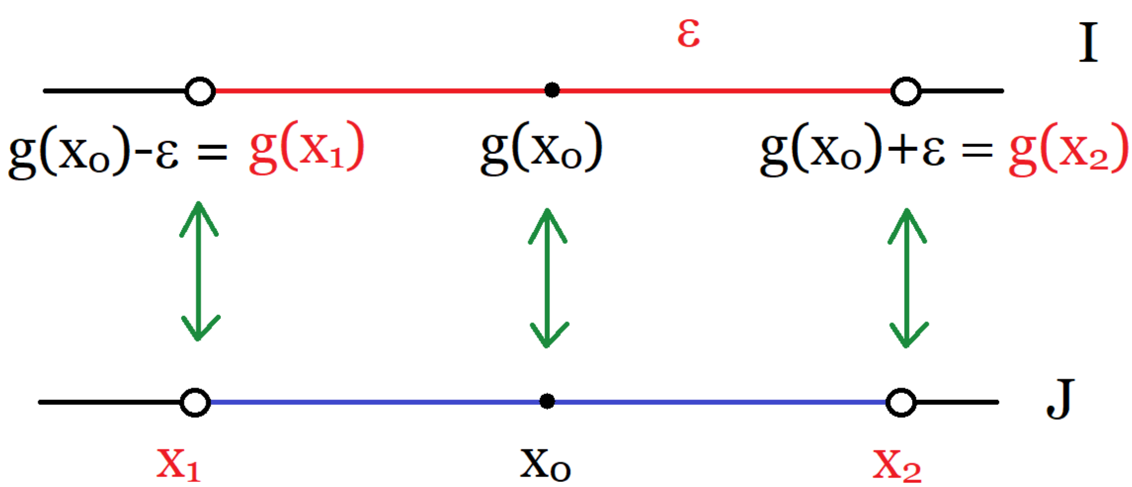
Then, since $\epsilon < r$ and by STEP 1, we have:

$$[g(x_0) - \epsilon, g(x_0) + \epsilon] \subseteq (g(x_0) - r, g(x_0) + r) \subseteq I$$



However, since $g(x_0) - \epsilon$ and $g(x_0) + \epsilon$ are in I and $g : J \to I$ is onto (since g is invertible), there are x_1 and x_2 in J such that $g(x_0) - \epsilon = g(x_1)$ and $g(x_0) + \epsilon = g(x_2)$





Claim: $x_1 < x_0 < x_2$

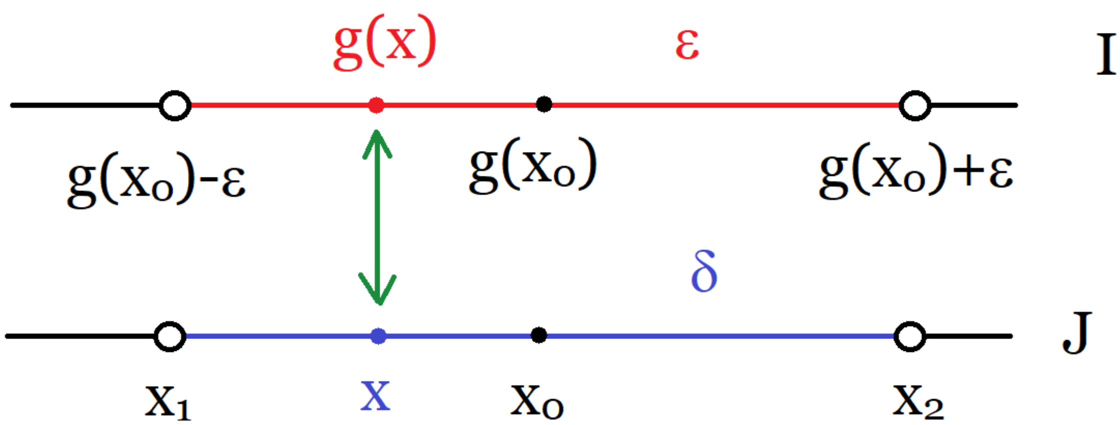
(This is at least true from the picture)

This is because

$$\begin{aligned}
&g(x_0) - \epsilon < g(x_0) < g(x_0) + \epsilon \\
\Rightarrow &g(x_1) < g(x_0) < g(x_2) \\
\Rightarrow &f(g(x_1)) < f(g(x_0)) < f(g(x_2)) \quad \text{(Since } f \text{ is increasing)} \\
\Rightarrow &x_1 < x_0 < x_2 \checkmark \quad \text{(Since } g = f^{-1}\text{)}
\end{aligned}$$

STEP 3: Actual Proof

With ϵ as above, let $\delta = \min\{|x_2 - x_0|, |x_1 - x_0|\}$



Intuition: With this δ, any x that is $\delta-$close to x_0 is guaranteed to be in the blue region above, and therefore $g(x)$ is guaranteed to be in the red/good region, so $g(x)$ will be $\epsilon-$close to $g(x_0)$

Then if $|x - x_0| < \delta$, then by definition of δ, we get $x_1 < x < x_2$ and therefore

$$
\begin{aligned}
&x_1 < x < x_2 \\
\Rightarrow & g(x_1) < g(x) < g(x_2) \quad \text{(Since } g \text{ is increasing)} \\
\Rightarrow & g(x_0) - \epsilon < g(x) < g(x_0) + \epsilon \quad \text{(By definition of } x_1 \text{ and } x_2\text{)} \\
\Rightarrow & -\epsilon < g(x) - g(x_0) < \epsilon \\
\Rightarrow & |g(x) - g(x_0)| < \epsilon
\end{aligned}
$$

Therefore $g = f^{-1}$ is continuous at x_0 □