LECTURE 19: UNIF. CONTINUITY (II) + LIMITS (I)

1. Uniform Continuity and Derivatives

Video: Uniform Continuity and Derivatives

Here is a nice trick for checking for uniform continuity:

Fact:

Suppose f' is bounded on (a,b), that is: there is $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (a,b)$.

Then f is uniformly continuous on (a,b)

Note: The same trick works for any interval, even infinite ones.

Example 1:

Let $f(x) = \frac{1}{x}$ on $(2,\infty)$ (continuous). Then $f'(x) = -\frac{1}{x^2}$ and therefore, for all $x \in (2,\infty)$ have

$$|f'(x)| = \left|-\frac{1}{x^2}\right| = \frac{1}{x^2} \leq \frac{1}{2^2} = \frac{1}{4} = M$$

Therefore f is uniformly continuous on $(2,\infty)$

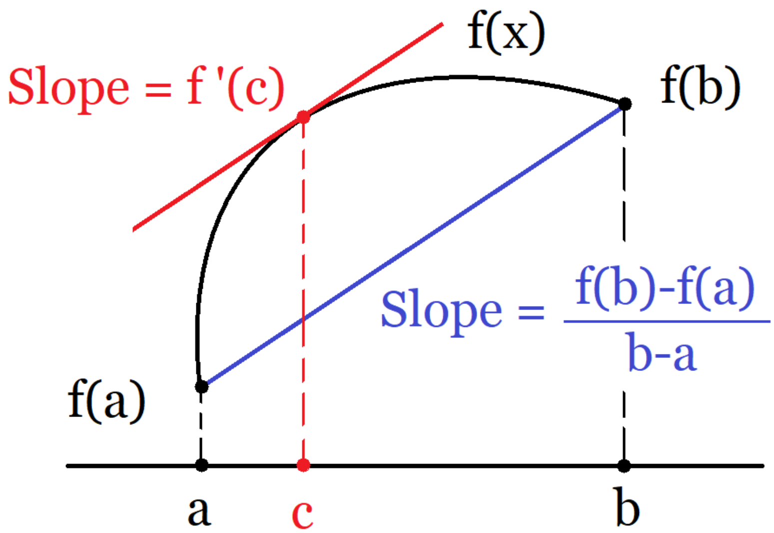
Date: Tuesday, November 2, 2021.

The proof of this uses the Mean Value Theorem, which we'll cover in Chapter 5

Mean Value Theorem:

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



Proof of Fact: Suppose $|f'(x)| \leq M$ for all x

Let $\epsilon > 0$ be given and let $\delta = \frac{\epsilon}{M}$

Then if $x, y \in (a, b)$ and $|x - y| < \delta$, by the Mean Value Theorem, there is c between x and y such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) \Rightarrow f(y) - f(x) = f'(c)(y - x)$$

But then $|f(y) - f(x)| = \underbrace{|f'(c)|}_{\leq M} |y - x| \leq M|y - x| < M\left(\frac{\epsilon}{M}\right) = \epsilon$✓

Hence f is uniformly continuous on (a, b) □

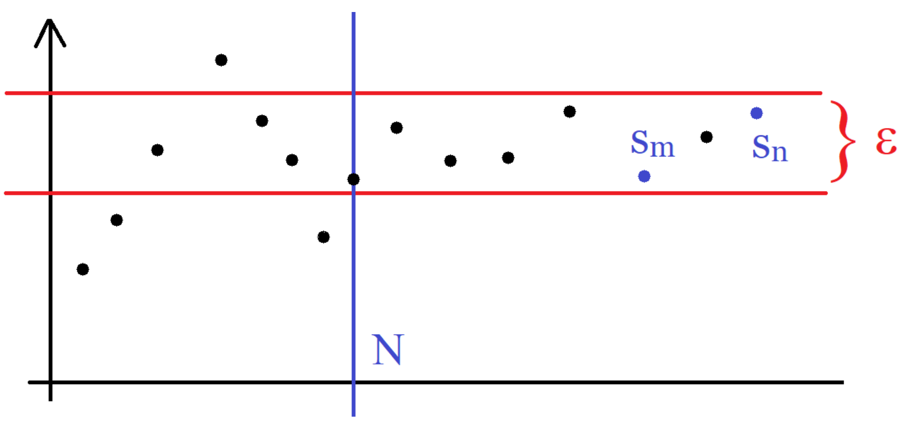
2. Uniform Continuity and Cauchy

Video: Uniform Continuity and Cauchy

Let's now discuss a useful property that helps us understand how uniformly continuous behave.

Recall (Section 10):

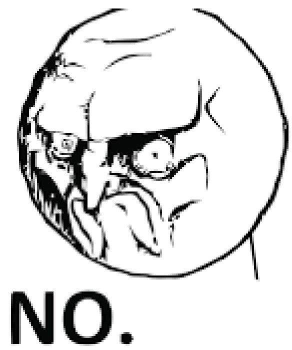
(s_n) is **Cauchy** if for all $\epsilon > 0$ there is N such that if $m, n > N$, then $|s_m - s_n| < \epsilon$



If f is continuous and (s_n) converges x_0, then, by definition, $f(s_n)$ is converges to $f(x_0)$

But what if we replace "converges" by "Cauchy" ?

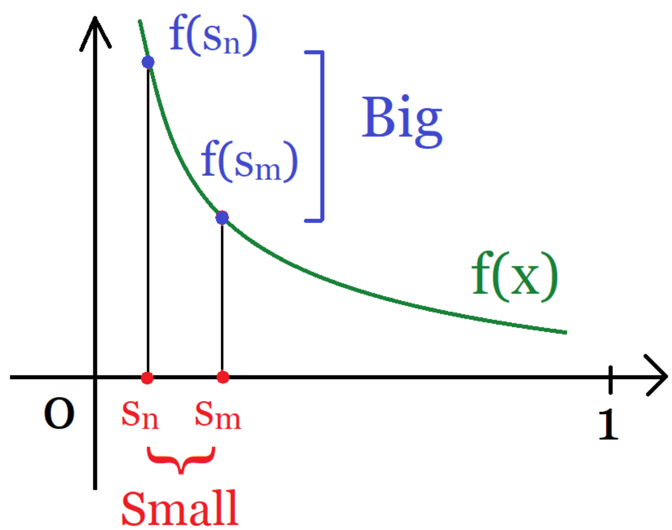
Question: If (s_n) is Cauchy and f is continuous, is $f(s_n)$ Cauchy?



Example:

Let $f(x) = \frac{1}{x}$ on $(0, 1)$

Then $s_n = \frac{1}{n}$ is Cauchy (because it converges), but $f(s_n) = \frac{1}{s_n} = n$ is not Cauchy (it doesn't even converge)

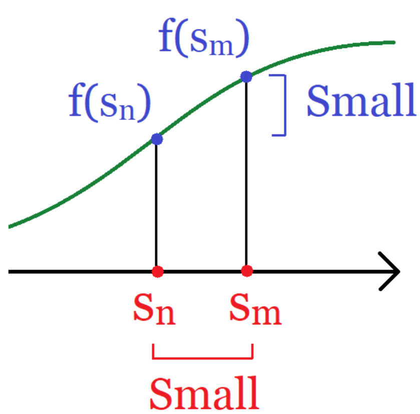


The reason this fails is because f is not *uniformly* continuous. And in fact, if f is uniformly continuous, then the answer is **YES**:

Fact:

If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $f(s_n)$ is Cauchy as well

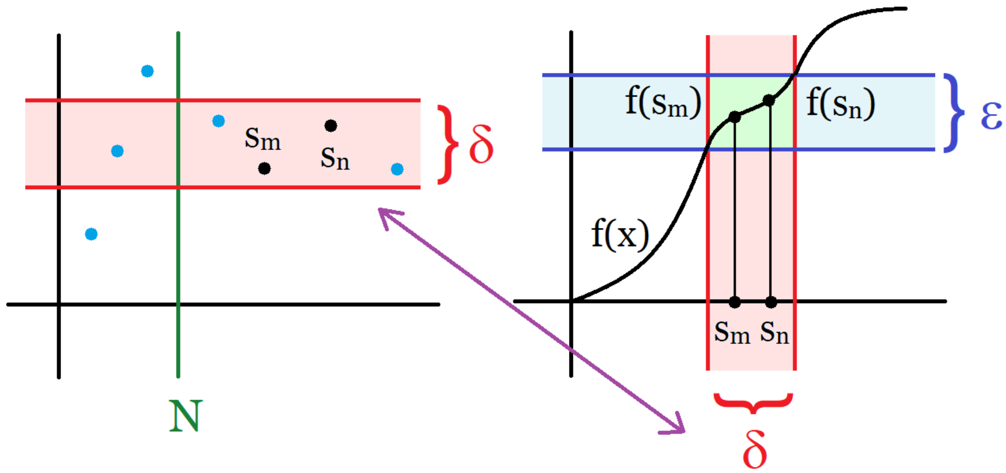
In other words, uniformly continuous functions take Cauchy sequences to Cauchy sequences.



Proof: Suppose (s_n) is Cauchy and let $\epsilon > 0$ be given. Since f is uniformly continuous on S, there is $\delta > 0$ such that if $x, y \in S$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$

Since (s_n) is Cauchy (with δ instead of ϵ), there is N such that if $m, n > N$, then $|s_n - s_m| < \delta$, and therefore $|f(s_n) - f(s_m)| < \epsilon$

Hence $f(s_n)$ is Cauchy $\square$



Note: This proof works *precisely* because f is uniformly continuous. Uniform continuity doesn't care about the precise location of the s_n. All we know is that the s_n are close to each other, which enough to conclude that $f(s_n)$ are close to each other.

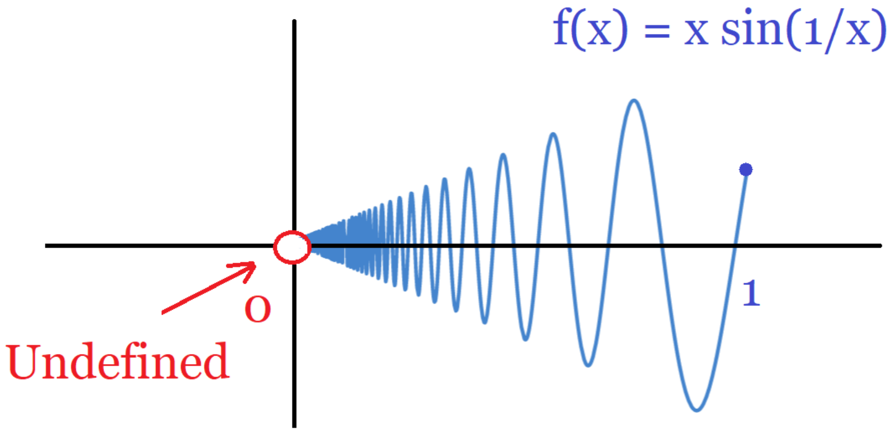
3. Continuous Extensions

Video: Continuous Extensions

This last property is useful because it relates uniform continuity with continuous extensions, something much more concrete.

Example 1:

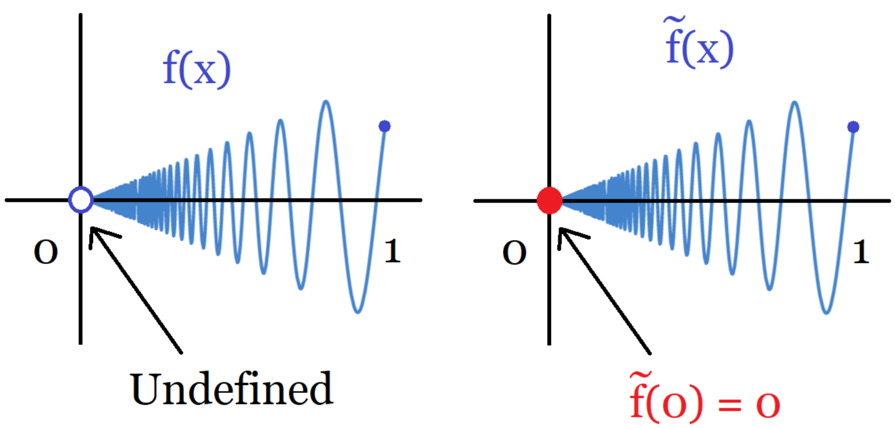
Let $f(x) = x \sin\left(\frac{1}{x}\right)$ on $(0, 1]$ (notice f is undefined at 0)



Problem: Can we define f at 0 to make it continuous at 0?

YES, just let $f(0) = 0$. In other words, if you let

$$\tilde{f}(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$$



(1) Then $\tilde{f}$ is continuous on $[0,1]$, and

(2) For $x \in (0,1]$, $\tilde{f}(x) = f(x)$

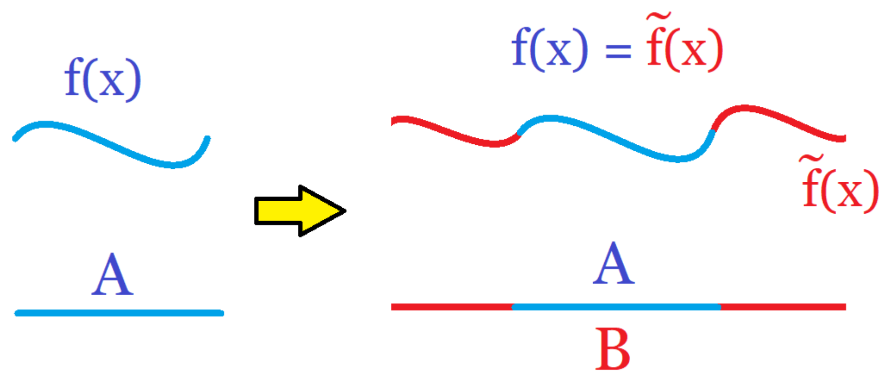
We call $\tilde{f}$ a **continuous extension** of f:

Definition:

Suppose $A \subseteq B$ and $f : A \to \mathbb{R}$ is continuous. Then $\tilde{f} : B \to \mathbb{R}$ is a **continuous extension** of f if

(1) $\tilde{f}$ is continuous on B and

(2) For all $x \in A$ we have $\tilde{f}(x) = f(x)$



So next time you ask for an extension to an assignment, ask for a *continuous* extension ☺

Fact:

Suppose $f : (a, b) \to \mathbb{R}$ is continuous. Then f is uniformly continuous if and only if it has a continuous extension $\tilde{f}$ on $[a, b]$

This is very useful for checking if a function is (or is not) uniformly continuous

Example 1:

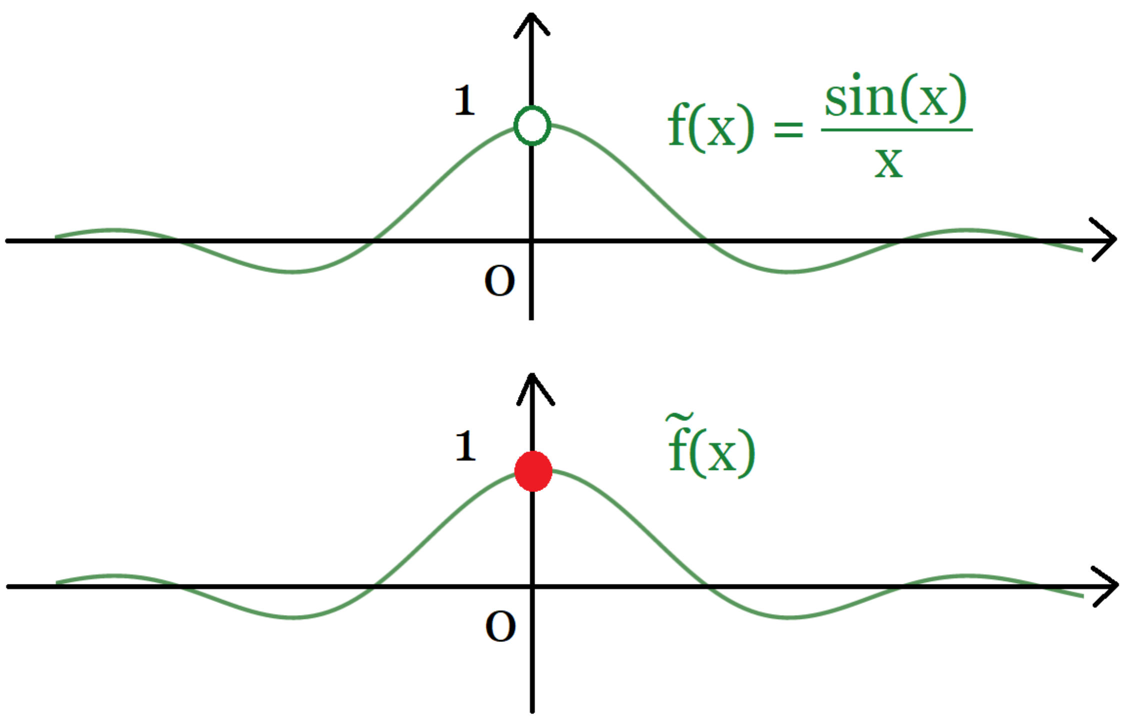
$f(x) = x \sin\left(\frac{1}{x}\right)$ is uniformly continuous on $(0,1]$ because it has a continuous extension $\tilde{f}(x)$

Example 2:

Let $f(x) = \frac{\sin(x)}{x}$ for $x \neq 0$, then f is uniformly continuous on $[-1,0) \cup (0,1]$ because $\tilde{f} : [-1,1] \to \mathbb{R}$ defined by:

$$\tilde{f}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Is a continuous extension of f

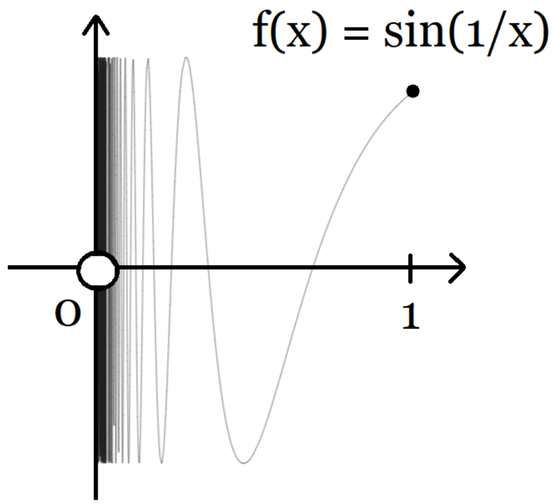


Note: The reason $\tilde{f}$ is continuous is because (from Calculus)

$$\lim_{x \to 0} \frac{\sin(x)}{x} \to 1$$

Example 3:

Let $f(x) = \sin\left(\frac{1}{x}\right)$ on $(0, 1]$.



Then f is not uniformly continuous on $(0, 1]$ because there is no continuous extension $\tilde{f}$: No matter how we define $\tilde{f}(0)$, $\tilde{f}$ will not be continuous on $[0, 1]$

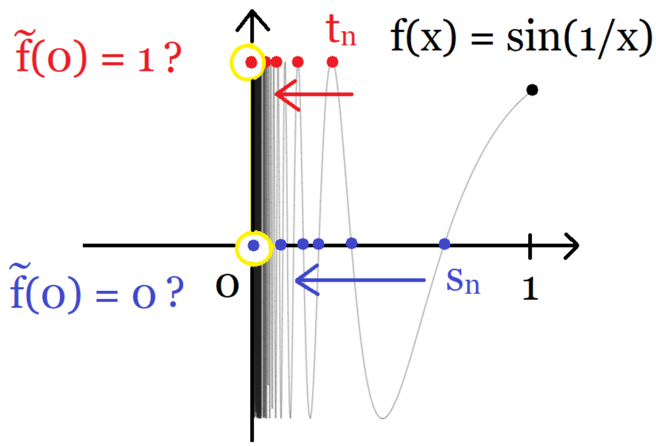
Why? Let $s_n = \frac{1}{\pi n} \to 0$. If $\tilde{f}$ were continuous at 0, then:

$$\tilde{f}(0) = \lim_{n \to \infty} \tilde{f}(s_n) = \lim_{n \to 0} f(s_n) = \sin\left(\frac{1}{s_n}\right) = \sin(\pi n) = 0$$

On the other hand, let $t_n = \frac{1}{\frac{\pi}{2} + 2\pi n} \to 0$. Then

$$\tilde{f}(0) = \lim_{n \to \infty} \tilde{f}(t_n) = \lim_{n \to 0} f(t_n) = \sin\left(\frac{1}{t_n}\right) = \sin\left(\frac{\pi}{2} + 2\pi n\right) = 1$$

Which contradicts $\tilde{f}(0) = 0 \Rightarrow\Leftarrow$. Hence $\tilde{f}$ cannot exist ☐



4. Limits of Functions

The nice thing about the definition of continuity is that it generalizes quite easily to limits.

Definition 1:

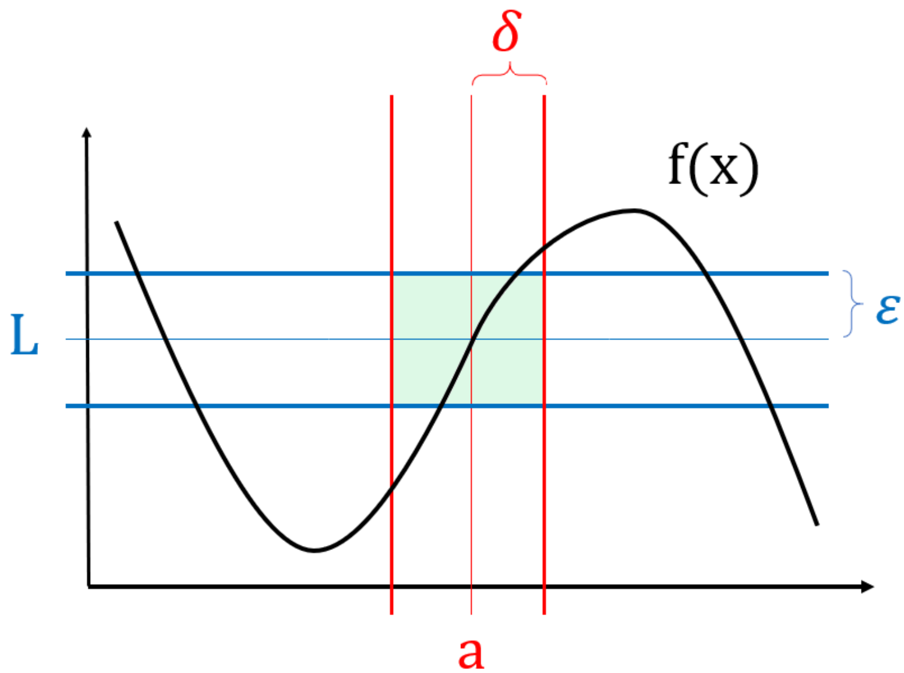
We say $\lim_{x\to a} f(x) = L$ if: whenever $x_n \to a$, then $f(x_n) \to L$

Definition 2:

We say $\lim_{x\to a} f(x) = L$ if: for all $\epsilon > 0$ there is $\delta > 0$ such that for all x, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$

Note: $0 < |x - a|$ just means that $x \neq a$, because limits don't care about what happens exactly at a

The two definitions are equivalent, with an almost identical proof to before.



5. Example 1: The Basics

Video: Example 1: The Basics

As an illustration, let's prove the following limit:

Example 1:

$$\text{Show: } \lim_{x\to 2} x^3 = 8$$

Sequence Definition: If $x_n \to 2$, then $(x_n)^3 \to 2^3 = 8$ ✓

Epsilon-Delta Definition:

STEP 1: Scratchwork

Show: for all $\epsilon > 0$ there is $\delta > 0$ such that if $0 < |x-2| < \delta$ then $\left|x^3 - 8\right| < \epsilon$.

$$\left|x^3 - 8\right| = \left|x^3 - 2^3\right| = |x-2|\left|x^2 + 2x + 4\right| < \epsilon$$

Here we used: $A^3 - B^3 = (A-B)(A^2 + AB + B^2)$

But if $|x-2| < 1$, then:

$$|x| = |x - 2 + 2| \leq |x-2| + |2| < 1 + 2 = 3$$

And so: $\left|x^2 + 2x + 4\right| \leq |x|^2 + 2|x| + 4 < (3)^2 + 2(3) + 4 = 9 + 6 + 4 = 19$

Therefore: $\left|x^3 - 8\right| = |x-2|\left|x^2 + 2x + 4\right| \leq |x-2|(19) < \epsilon \Rightarrow |x-2| < \dfrac{\epsilon}{19}$

This suggests to let $\delta = \frac{\epsilon}{19}$, but also remember that we assumed $|x-2| < 1$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $\delta = \min\left\{\frac{\epsilon}{19}, 1\right\}$, then if $0 < |x-2| < \delta$, then $|x-2| < 1$, so $\left|x^2 + 2x + 4\right| < 19$, but also $|x-2| < \frac{\epsilon}{19}$, and so:

$$\left|x^3 - 8\right| = |x-2|\left|x^2 + 2x + 4\right| \leq |x-2|(19) < \left(\frac{\epsilon}{19}\right)(19) = \epsilon\checkmark$$

Hence $\lim_{x\to 2} x^3 = 8$

Note: For more practice with limits, check out the following videos:

Video 1: Linear Function

Video 2: Squares

Video 3: Square Root

Video 4: Reciprocals

6. Example 2: Infinite Limit at a Point

Video: Example 2: Infinite Limit at a point

In this example, I'll cover both a one-sided limit, and an infinite limit at a point:

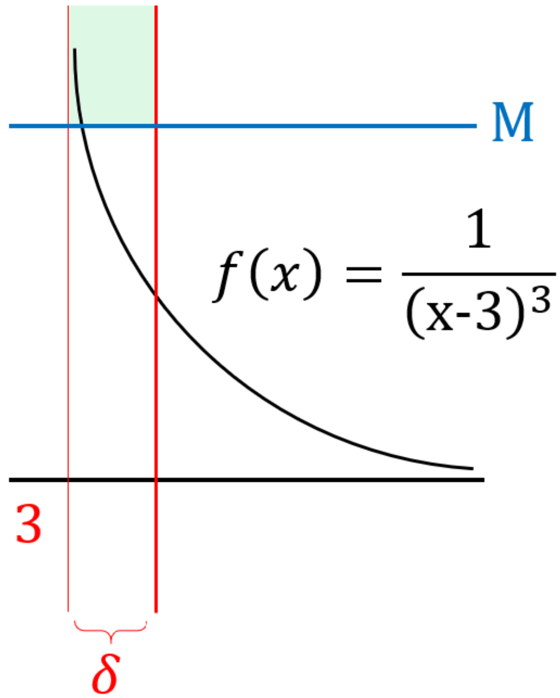
Example 2:

$$\text{Show: } \lim_{x \to 3^+} \frac{1}{(x-3)^3} = \infty$$

$x \to 3^+$ just means the limit as x approaches 3 from the right.

Note: For $x \to 3^+$, just replace $|x-3|$ by $x-3$, and for $x \to 3^-$, just replace $|x-3|$ by $-(x-3) = 3-x$

Here we just want to say: No matter how big a number M, we can make $\frac{1}{(x-3)^3}$ bigger than M by making x close enough to 3:



STEP 1: Scratchwork

Show: For all $M > 0$ there is $\delta > 0$ such that if $0 < (x-3) < \delta$, then $\frac{1}{(x-3)^3} > M$

$$\text{But: } \frac{1}{(x-3)^3} > M \Rightarrow (x-3)^3 < \frac{1}{M} \Rightarrow x - 3 < \sqrt[3]{\frac{1}{M}}$$

Which suggests to let $\delta = \frac{1}{\sqrt[3]{M}}$

STEP 2: Actual Proof

Let $M > 0$ be given, let $\delta = \frac{1}{\sqrt[3]{M}}$, then if $0 < x - 3 < \delta$, then

$$\frac{1}{(x-3)^3} > \frac{1}{\delta^3} = \frac{1}{\frac{1}{M}} = M \checkmark$$

Hence $\lim_{x\to\infty} \frac{1}{(x-3)^3} = \infty$

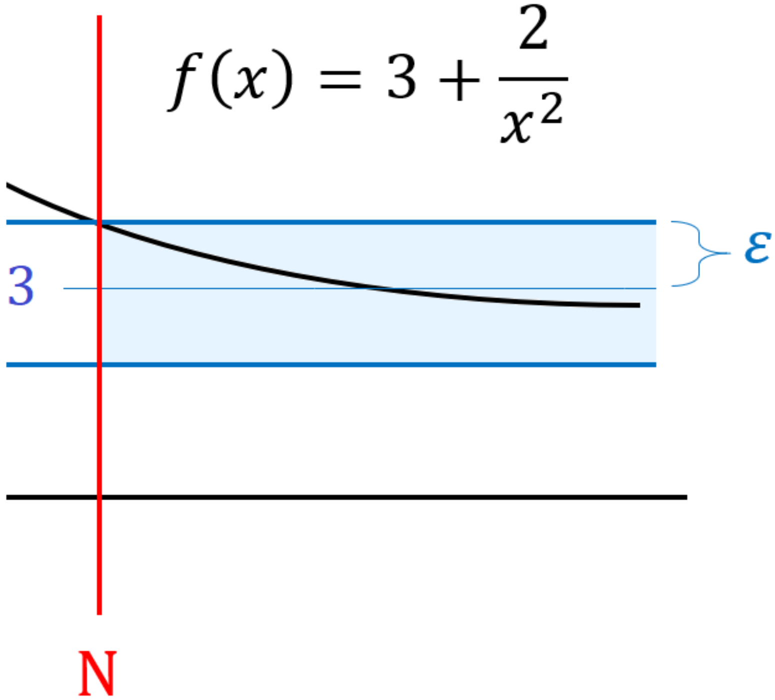
7. Example 3: Limits at Infinity

Video: Example 3: Limit at infinity

Pretty much identical to the sequence definition from section 8:

Example 3:

$$\text{Show: } \lim_{x\to\infty} 3 + \frac{2}{x^2} = 3$$



STEP 1: Scratchwork

Show: For all $\epsilon > 0$ there is N such that if $x > N$ then $\left|3 + \frac{2}{x^2} - 3\right| < \epsilon$

$$\text{But: } \left|3 + \frac{2}{x^2} - 3\right| = \frac{2}{x^2} < \epsilon \Rightarrow x^2 > \frac{}{2\epsilon} \Rightarrow x > \frac{1}{\sqrt{2\epsilon}}$$

Which suggests to let $N = \frac{1}{\sqrt{2\epsilon}}$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $N = \frac{1}{\sqrt{2\epsilon}}$, then if $x > N$, then

$$\left|3 + \frac{2}{x^2} - 3\right| = \frac{2}{x^2} < \frac{2}{\frac{2}{\epsilon}} = \epsilon \checkmark$$

Note: For $\lim_{x\to-\infty} f(x)$, we replace $x > N$ with $x < N$.

Note: We can also define $\lim_{x\to\infty} f(x) = \infty$: For all $M > 0$ there is N such that if $x > N$ then $f(x) > M$.

8. Optional: Proof of Continuous Extension

Fact:

Suppose $f : (a, b) \to \mathbb{R}$ is continuous. Then f is uniformly continuous if and only if it has a continuous extension $\tilde{f}$ on $[a, b]$

Proof: ($\Leftarrow$) By definition $\tilde{f}$ is continuous on $[a, b]$, so, by the fact from last time, $\tilde{f}$ is uniformly continuous on $[a, b]$, so $f = \tilde{f}$ is uniformly continuous on the smaller interval (a, b)

($\Rightarrow$) The proof is magical! We'll do some wishful thinking that actually works.

STEP 1: Suppose f is uniformly continuous on (a,b). Since on (a,b), $\tilde{f}(x) =: f(x)$ is continuous, all we really need to do is define $\tilde{f}(a)$ and show $\tilde{f}$ is continuous at a (the case $\tilde{f}(b)$ is similar)

Main Idea:

If $\tilde{f}$ were continuous at a, **then** for any sequence (s_n) in (a,b) with $s_n \to a$, we would have

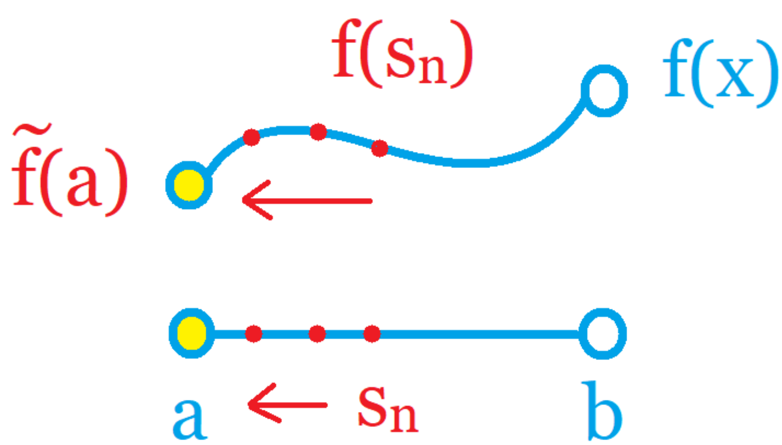
$$\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} \tilde{f}(s_n) = \tilde{f}(a)$$

(Here we used $s_n \in (a,b)$ and $\tilde{f} = f$ on (a,b))

The idea is then to *define* $\tilde{f}(a)$ as:

$$\tilde{f}(a) =: \lim_{n\to\infty} f(s_n)$$

Where (s_n) is any sequence in (a,b) converging to a



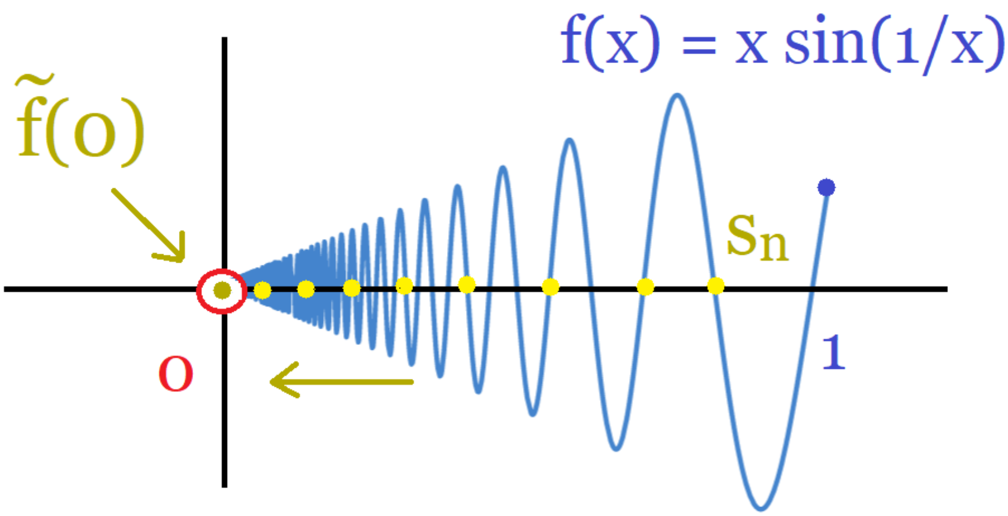
Example:

Take again $f(x) = x \sin\left(\frac{1}{x}\right)$. What is $\tilde{f}(0)$?

Let $s_n = \frac{1}{\pi n} \to 0$. Then, by the above, we have

$$\tilde{f}(0) = \lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} s_n \sin\left(\frac{1}{s_n}\right) = \lim_{n\to\infty}\left(\frac{1}{\pi n}\right)\underbrace{\sin(\pi n)}_{0} = 0$$

Therefore $\tilde{f}(0) = 0$



The definition above seems too good to be true! We're *literally* defining $\tilde{f}(a)$ in such a way that it solves our problem. It turns out that it actually works. But in order to make sure that $\tilde{f}(a)$ is well-defined, we need to answer the following questions:

(1) Does $f(s_n)$ even converge? (otherwise $\lim f(s_n)$ makes no sense)

(2) More importantly: Is the above limit independent of the choice of the sequence (s_n) used?

STEP 2:

Claim 1: If (s_n) is a sequence in (a, b) that converges to a, then $f(s_n)$ converges

Proof of Claim 1: Since (s_n) converges, (s_n) is Cauchy, and therefore, since f is uniformly continuous, by the previous section, $f(s_n)$ is Cauchy, and therefore $f(s_n)$ converges ✓

STEP 3:

Claim 2: Suppose (s_n) and (t_n) are two sequences in (a, b) converging to a, then

$$\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} f(t_n)$$

(This shows that the definition $\tilde{f}(a)$ above does not depend on the choice of (s_n))

Proof of Claim 2: Suppose (s_n) and (t_n) both converge to a.

Here's a neat idea: let's *interlace* the two sequences (s_n) and (t_n) to get a new sequence (u_n):

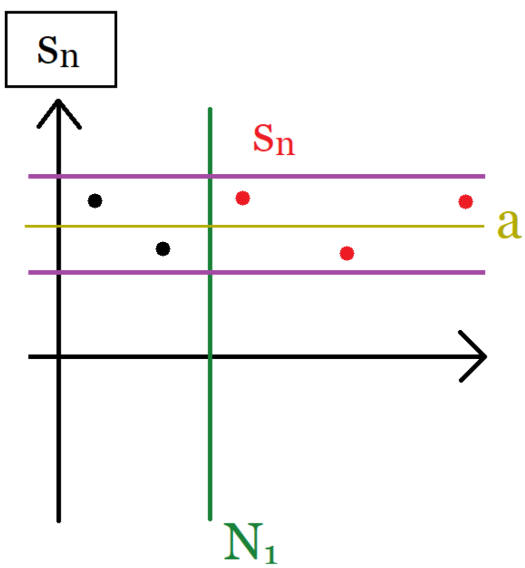
$$(u_n) = (s_1, t_1, s_2, t_2, \dots)$$

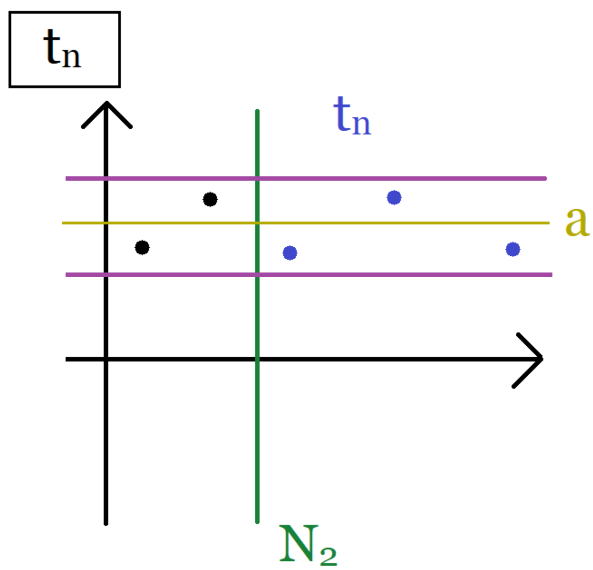
Claim 3: (u_n) converges to a

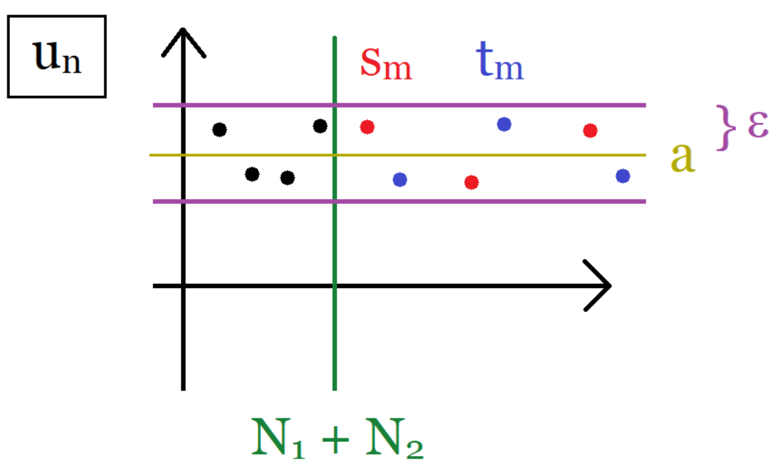
Proof of Claim 3:

Let $\epsilon > 0$ be given.

Since $s_n \to a$, there is N_1 such that if $n > N_1$, then $|s_n - a| < \epsilon$, and since $t_n \to a$, there is N_2 such that if $n > N_2$, then $|t_n - a| < \epsilon$.







Let $N = N_1 + N_2$

Then if $n > N$, either $u_n = s_m$ for some $m > N_1$ in which case $|u_n - a| = |s_m - a| < \epsilon$; or $u_n = t_m$ for some $m > N_2$, in which case $|u_n - a| = |t_m - a| < \epsilon$ as well ✓

Since $u_n \to a$ and f is continuous,

$$f(u_n) = (f(s_1), f(t_1), f(s_2), f(t_2), \dots)$$

converges to some $s \in \mathbb{R}$. Therefore, any subsequence of $f(u_n)$ converges to s as well.

But $f(s_n) = (f(s_1), f(s_2), \dots)$ is a subsequence of $f(u_n)$, and hence converges to s. Similarly $f(t_n) = (f(t_1), f(t_2), \dots)$ is a subsequence of $f(u_n)$, hence converges to s as well.

$$\text{Therefore } \lim_{n\to\infty} f(s_n) = s = \lim_{n\to\infty} f(t_n) ✓$$

STEP 4: Define:

$$\tilde{f}(a) =: \lim_{n\to\infty} f(s_n)$$

Where (s_n) is any sequence in (a, b) converging to a

By **STEP 2** and **STEP 3**, $\tilde{f}(a)$ is well-defined.

It is enough to check that $\tilde{f}$ is continuous at $x = a$

Let (s_n) be a sequence in $[a, b]$ converging to a, we need to show $\tilde{f}(s_n) \to \tilde{f}(a)$

WLOG, assume $s_n \in (a,b)$, so (s_n) is a sequence in (a,b) converging to a, and therefore:

$$\lim_{n\to\infty} \tilde{f}(s_n) = \lim_{n\to\infty} f(s_n) = \tilde{f}(a) \checkmark$$

In the first step, we used $s_n \in (a,b)$ and in the second step we used the DEFINITION of $\tilde{f}(a)$

Hence $\tilde{f}$ is a continuous extension of f ✓ □