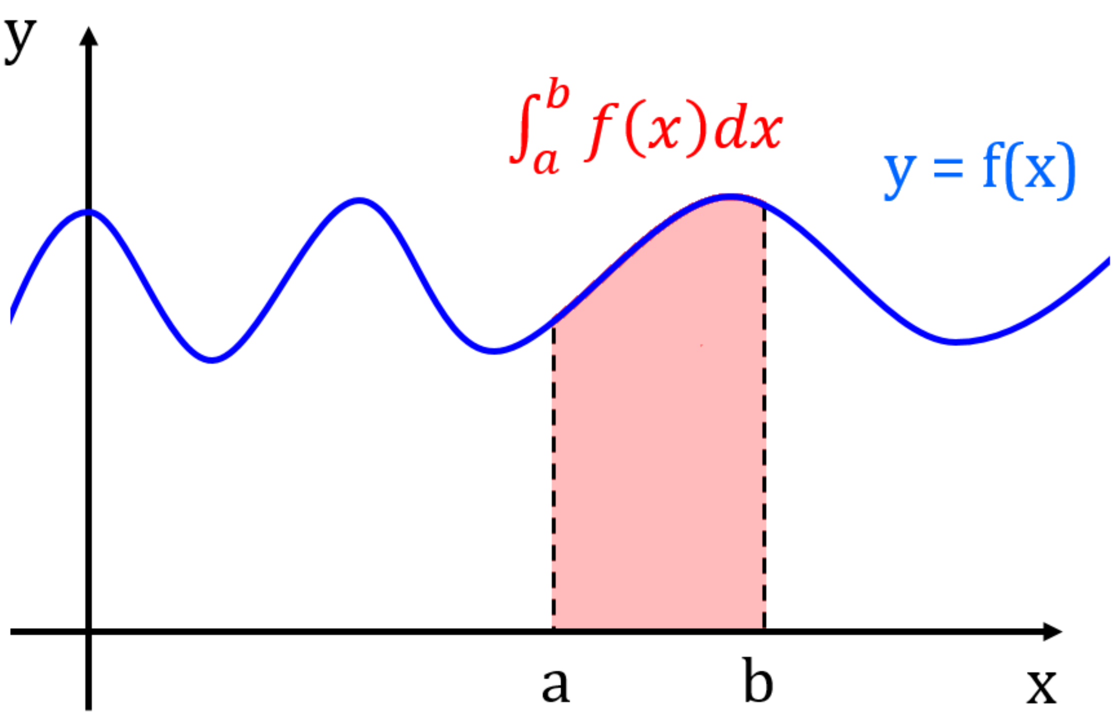
LECTURE 19: DOUBLE INTEGRALS (I)

Welcome to the world of double integrals! It's double the integrals, but triple the fun!

1. Introduction

Recall:

$\int_a^b f(x)dx$ represents the **area** under the graph of f from a to b



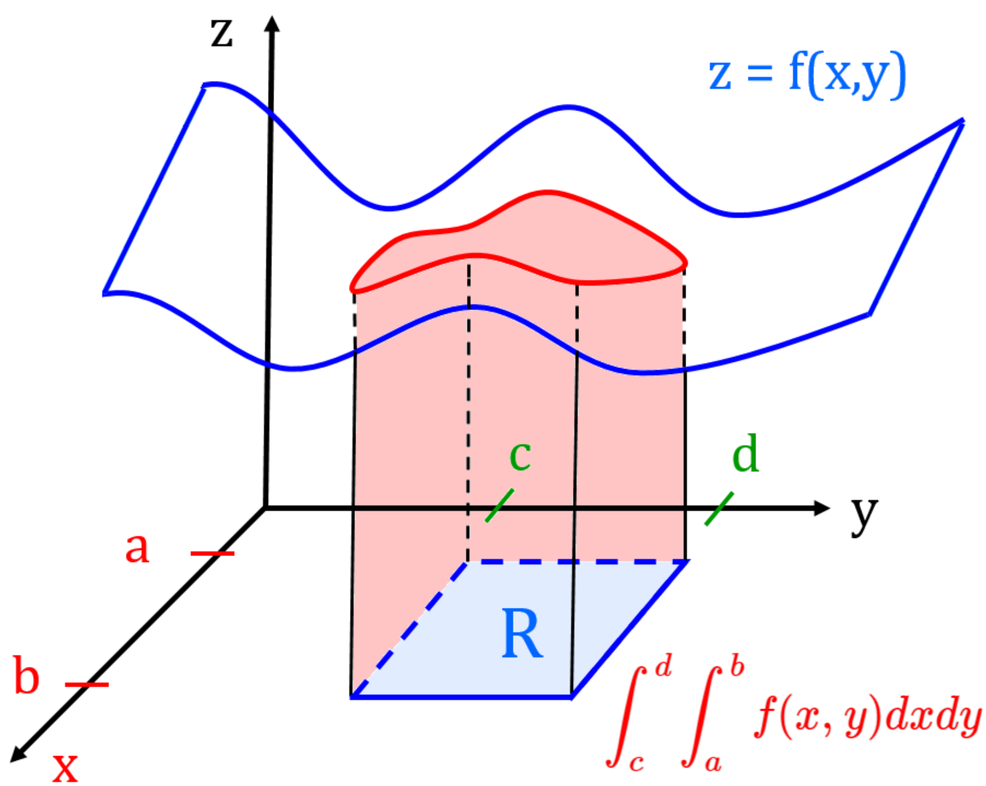
Date: Monday, October 11, 2021.

In three dimensions, it's the same, except areas becomes volumes, and integrals become double integrals:

Double Integral:

$$\int_c^d \int_a^b f(x,y)dxdy = \textbf{VOLUME} \text{ under the graph of } f$$

$$\text{and over the rectangle } R = [a,b] \times [c,d]$$



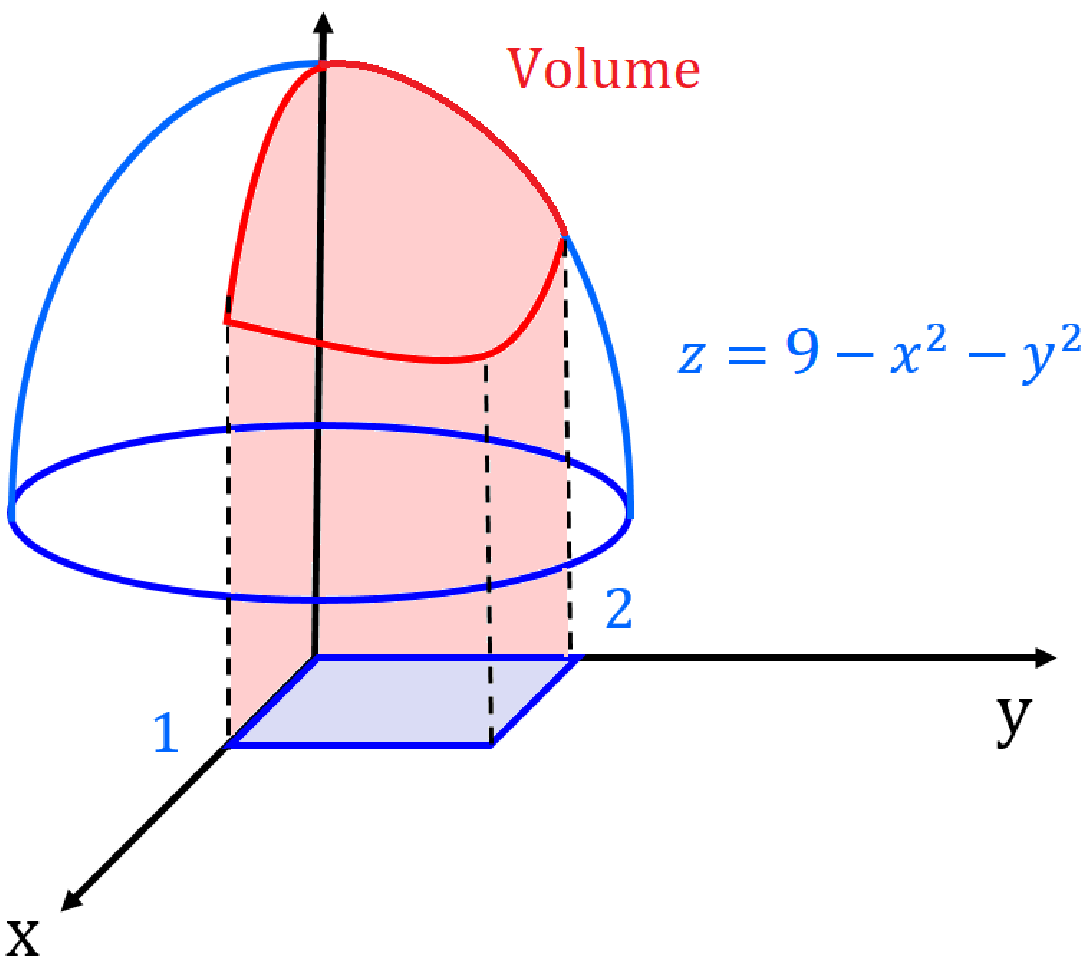
Analogy: Think of slicing a cake, where the base is a rectangle and the height is your function.

2. Calculating Double Integrals

This is the definition, but do we actually calculate double integrals?

Example 1:

Calculate the volume under the graph of $f(x, y) = 9 - x^2 - y^2$ and over $[0, 1] \times [0, 2]$



Here x is between 0 and 1, and y is between 0 and 2.

Note: The technique is the same as for derivatives, except here you integrate with respect to x (treating y as a constant) instead of differentiate with respect to x.

$$
\begin{aligned}
\text{Volume } &= \int_0^2 \int_0^1 9 - x^2 - y^2 dx dy \\
&= \int_0^2 \left(\int_0^1 9 - x^2 - y^2 dx \right) dy \\
&= \int_0^2 \left[9x - \frac{x^3}{3} - y^2 x \right]_{x=0}^{x=1} dy \\
&= \int_0^2 9(1) - \frac{(1)^3}{3} - y^2(1) - 9(0) + \frac{(0)^3}{3} - y^2(0) dy \\
&= \int_0^2 9 - \frac{1}{3} - y^2 dy \\
&= \int_0^2 \frac{26}{3} - y^2 dy \\
&= \left[\frac{26}{3} y - \frac{y^3}{3} \right]_0^2 \\
&= \left(\frac{26}{3} \right) (2) - \frac{(2)^3}{3} - \left(\frac{26}{3} \right) (0) + \frac{0^3}{3} \\
&= \frac{52}{3} - \frac{8}{3} \\
&= \frac{44}{3}
\end{aligned}
$$

Notice how we turned the double integral (with x and y) into a single-variable integral (with y only). This idea will be extremely important when we'll do triple integrals.

Example 2:

(a) Calculate

$$\int_1^2 \int_{-3}^1 x^2 y dx dy$$

$$\begin{aligned}
&\int_1^2 \left(\int_{-3}^1 x^2 y \, dx \right) dy \\
=&\int_1^2 \left[\left(\frac{x^3}{3} \right) y \right]_{x=-3}^{x=1} dy \\
=&\int_1^2 \frac{(1)^3}{3} y - \frac{(-3)^3}{3} y \, dy \\
=&\int_1^2 \left(\frac{1}{3} \right) y + 9y \, dy \\
=&\int_1^2 \left(\frac{28}{3} \right) y \, dy \\
=&\left[\left(\frac{28}{3} \right) \frac{y^2}{2} \right]_1^2 \\
=&\left[\left(\frac{14}{3} \right) y^2 \right]_1^2 \\
=&\left(\frac{14}{3} \right) (4-1) \\
=&\left(\frac{14}{3} \right) (3) \\
=&14
\end{aligned}$$

(b) Calculate

$$\int_{-3}^{1}\int_{1}^{2} x^2 y \, dy dx$$

Here x and y are switched, so you do y first:

$$\begin{aligned}
\text{Answer} &= \int_{-3}^{1}\left(\int_{1}^{2} x^2 \, y \, dy\right) dx \\
&= \int_{-3}^{1}\left[x^2\left(\frac{y^2}{2}\right)\right]_{y=1}^{y=2} dx \\
&= \int_{-3}^{1} x^2\left(\frac{4}{2}\right) - x^2\frac{1}{2} dx \\
&= \int_{-3}^{1}\left(\frac{3}{2}\right) x^2 dx \\
&= \left[\left(\frac{3}{2}\right)\left(\frac{x^3}{3}\right)\right]_{-3}^{1} \\
&= \left[\left(\frac{x^3}{2}\right)\right]_{-3}^{1} \\
&= \frac{1}{2}\left(1-(-3)^3\right) \\
&= \frac{1}{2}(1+27) \\
&= \frac{28}{2} \\
&= 14
\end{aligned}$$

Notice: We have the same answer! In fact this is always true, and has a special name:

3. Fubini's Theorem

Fubini's Theorem:

$$\int_c^d \int_a^b f(x,y)\, dxdy = \int_a^b \int_c^d f(x,y)\, dydx$$

Or, in layman's terms, $dxdy = dydx$. Compare with Clairaut, which says $f_{xy} = f_{yx}$

In other words, the order of integration doesn't matter; we can evaluate a double integral in any order that we want (either x first and then y or y first and then x).

Note: Strictly speaking, this theorem isn't *always* true (but will be true in our class). You can check out the following optional video for an interesting counterexample:

Video: Fubini Counterexample (optional)

Definitely use this theorem to your advantage, it will simplify your calculations tremendously:

Example 3:

$$\int_0^\pi \int_1^2 x\cos(xy)dxdy$$

First Try:

$$\int_0^\pi \int_1^2 x\cos(xy)dxdy = \int_0^\pi \left[\ ?\ \right]_{x=1}^{x=2} dy$$

Where $\boxed{?}$ is an anti-derivative of $x\cos(xy)$ with respect to x

Too hard! *Could* integrate by parts, but ain't nobody got time for that!

Second Try: Use Fubini!

$$\begin{aligned}
\int_0^\pi \int_1^2 x\cos(xy)\,dx dy &= \int_1^2 \int_0^\pi x\cos(xy) dy dx \qquad (\text{Think } 3\cos(3y)) \\
&= \int_1^2 \left[x \left(\frac{\sin(xy)}{x} \right) \right]_{y=0}^{y=\pi} dx \\
&= \int_1^2 [\sin(xy)]_{y=0}^{y=\pi} dx \\
&= \int_1^2 \sin(x\pi) - \sin(x0) dx \\
&= \int_1^2 \sin(\pi x) dx \\
&= \left[\frac{-\cos(\pi x)}{\pi} \right]_1^2 \\
&= -\left(\frac{\cos(2\pi)}{\pi} \right) + \frac{\cos(\pi)}{\pi} \\
&= -\frac{1}{\pi} - \frac{1}{\pi} \\
&= -\frac{2}{\pi}
\end{aligned}$$

4. Volume of a ravioli

Video: Volume of a Ravioli

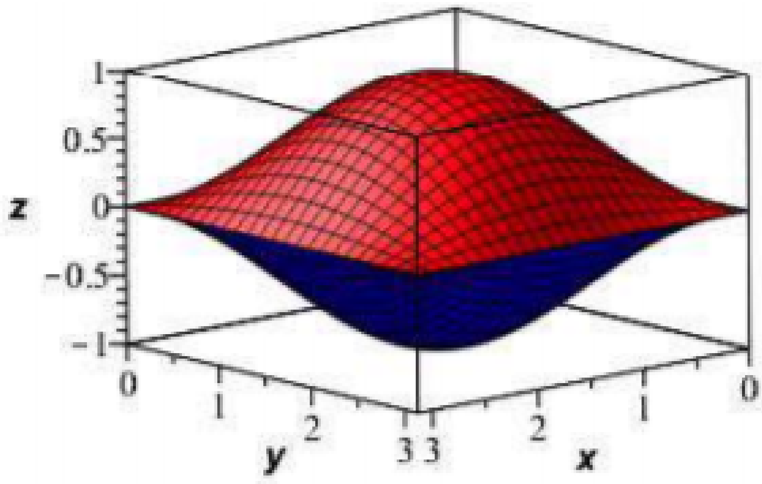
Finally, here is a nice trick for calculating integrals of the form $f(x)g(y)$:

Example 4:

Calculate the volume of a ravioli on $[0,\pi] \times \left[0, \frac{\pi}{2}\right]$ between the functions

$$z = \sin(x)\cos(y) \text{ and } z = -\sin(x)\cos(y)$$

The picture looks similar to this:



Here the volume between $-\sin(x)\cos(y)$ and $\sin(x)\cos(y)$ is given by

$$\int\int \underbrace{\sin(x)\cos(y)}_{\text{Bigger}} - \underbrace{(-\sin(x)\cos(y))}_{\text{Smaller}} = \int_0^{\frac{\pi}{2}} \int_0^{\pi} 2\sin(x)\cos(y)dxdy$$

Useful Trick:

$$\int_c^d \int_a^b f(x)g(y)dxdy = \left(\int_a^b f(x)dx\right)\left(\int_c^d g(y)dy\right)$$

Why?

$$\int_c^d \int_a^b f(x)g(y)\,dxdy = \int_c^d g(y) \underbrace{\left(\int_a^b f(x)dx\right)}_{\text{Constant}} dy = \left(\int_a^b f(x)dx\right)\left(\int_c^d g(y)dy\right)$$

Here:

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \int_0^{\pi} 2\sin(x)\cos(y)dxdy \\ =& 2\left(\int_0^{\pi} \sin(x)dx\right)\left(\int_0^{\frac{\pi}{2}} \cos(y)dy\right) \\ =& 2\left[-\cos(x)\right]_0^{\pi}\left[\sin(y)\right]_0^{\frac{\pi}{2}} \\ =& 2\left(-\cos(\pi)+\cos(0)\right)\left(\sin\left(\frac{\pi}{2}\right)-\sin(0)\right) \\ =& 2(1+1)(1-0) \\ =& 4 \end{aligned}$$

⚠ This ONLY works if a, b, c, d are constants, do NOT attempt this on integrals where the endpoints are variable, like:

$$\int_0^1 \int_{x^2}^{8-x^2} f(x)g(y)dydx \neq \left(\int_0^1 f(x)dx\right)\left(\int_{x^2}^{8-x^2} g(y)dy\right)$$