LECTURE 19: MEASURABLE FUNCTIONS

1. The Lebesgue Measure

Recall: E is **mesurable** if for every $\epsilon > 0$ there is an open set $O \supseteq E$ such that $m_\star(O - E) \leq \epsilon$

Definition: If E is measurable, then the **Lebesgue Measure** is

$$m(E) = m_\star(E) = \inf \sum_{j=1}^{\infty} |Q_j|$$

The inf is taken over all countable coverings $E \subseteq \bigcup_{j=1}^{\infty} Q_j$ with closed cubes.

Property 2: If $m_\star(E) = 0$ then E is measurable

Why? Let $\epsilon > 0$ be given. Then since $\epsilon > 0 = m_\star(E) = \inf m_\star(O)$ (here O ranges over all open sets containing E), by def of inf there is an open set O containing E with $m_\star(O) \leq \epsilon$ and hence $m_\star(O - E) \leq m_\star(O) \leq \epsilon$ ✓

Property 3: A countable union of measurable sets is measurable.

Let $E = \bigcup_{j=1}^{\infty} E_j$ where each E_j is measurable, and let $\epsilon > 0$ be given.

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For each j there is an open set $O_j \supseteq E_j$ with $m_\star(O_j - E_j) \leq \frac{\epsilon}{2^j}$. Then the union $O =: \bigcup_{j=1}^{\infty} O_j$ is open, $E \subseteq O$ and $O - E \subseteq \bigcup_{j=1}^{\infty} O_j - E_j$ and so by monotonicity and countable sub-additivity we get

$$m_\star(O - E) \leq \sum_{j=1}^{\infty} m_\star(O_j - E_j) \leq \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon \quad \square$$

Property 4: Closed sets F are measurable

Will skip the proof in lecture. For a partial proof, see Appendix below.

Property 5: The complement of a measurable set is measurable

If E is measurable, then for every n there is an open set $O_n \supseteq E$ with $m_\star(O_n - E) \leq \frac{1}{n}$. The complement O_n^c is closed, hence measurable, and so the union $S = \bigcup_{n=1}^{\infty} O_n^c$ is measurable. Now simply note that $S \subseteq E^c$ and for all n, we have

$$E^c - S \subseteq O_n - E$$

Therefore $m_\star(E^c - S) \leq m_\star(O_n - E) \leq \frac{1}{n}$ and since n was arbitrary we get $m_\star(E^c - S) = 0$ hence $E^c - S$ is measurable and therefore $E^c = (E^c - S) \cup S$ is measurable $\square$

Property 6: A countable intersection of measurable sets is measurable

$$\text{Follows from } \bigcap_{j=1}^{\infty} E_j = \left(\bigcup_{j=1}^{\infty} E_j^c \right)^c$$

Definition: If E is any set, then a collection $\mathcal{R}$ of subsets of E is called a σ**-algebra** if:

(1) If $A \in \mathcal{R}$ then $A^c \in \mathcal{R}$

(2) If $A_1, A_2, \cdots \in \mathcal{R}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$

(3) (If $A_1, A_2, \cdots \in \mathcal{R}$ then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$)

(Technically (3) follows from (1) and (2))

We have shown that the collection of Lebesgue measurable subsets of $\mathbb{R}^d$ forms a σ-algebra. It is a σ-algebra containing all open subsets of $\mathbb{R}^d$. The smallest such σ-algebra is called the **Borel $\sigma-$algebra** (and is even smaller)

Property 5: If $E_1, E_2, \ldots$ are disjoint **measurable sets** and $E = \bigcup_{j=1}^{\infty} E_j$ then

$$m(E) = \sum_{j=1}^{\infty} m(E_j)$$

For a proof of this, see Theorem 3.2 in Stein and Shakarchi. The idea is to find compact subsets F_j of E_j with $m_\star(E_j - F_j) < \frac{\epsilon}{2^j}$ (can do since E_j^c is measurable). And since $F_1, \cdots, F_N$ (for finite N) are compact and disjoint, they are a positive distance from each other, and let $N \to \infty$

2. Geometric Properties

Fact: Suppose $E \subseteq \mathbb{R}^d$ is measurable, then for every $\epsilon > 0$

(1) There is an open set $E \subseteq O$ such that $m(O - E) < \epsilon$

(2) There is a closed set $F \subseteq E$ with $m(E - F) \leq \epsilon$

(3) If $m(E) < \infty$ there is a finite union $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes with $m(E \Delta F) \leq \epsilon$

Here $E\Delta F = (E - F) \cup (F - E)$ is the set of points that belong to only one of the two sets F.

(1) follows from the definition and (2) follows since E^c is measurable

Proof of (3) Let $\epsilon > 0$ be given and choose a family of closed cubes $\{Q_j\}_{j=1}^{\infty}$ such that

$$E \subseteq \bigcup_{j=1}^{\infty} Q_j \text{ and } \sum_{j=1}^{\infty} |Q_j| \leq m(E) + \frac{\epsilon}{2}$$

Since $m(E) < \infty$ the series converges and so there is $N > 0$ such that $\sum_{j=N+1}^{\infty} |Q_j| < \frac{\epsilon}{2}$. If $F =: \bigcup_{j=1}^{N} Q_j$ then

$$\begin{aligned} m(E\Delta F) =& m((E - F) \cup (F - E)) \\ =& m(E - F) + m(F - E) \\ \leq& m\left(\bigcup_{j=N+1}^{\infty} Q_j\right) + m\left(\bigcup_{j=1}^{\infty} Q_j - E\right) \\ \leq& \sum_{j=N+1}^{\infty} |Q_j| + \left(\sum_{j=1}^{\infty} |Q_j| - m(E)\right) \\ <& \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ =& \epsilon \quad \square \end{aligned}$$

Invariance Properties:

(1) If $E_h = \{x + h \mid x \in E\}$ then $m(E_h) = m(E)$

(2) $m(-E) = m(E)$

(3) If $\delta E = \{\delta x \mid x \in E\}$ then $m(\delta E) = \delta^d m(E)$

This just follows because, for example, if O is open, so is O_h and $m(O_h - E_h) = m(O - E) < \epsilon$

3. A non-measurable set

Video: A non-measurable set

Here is an example of a non-measurable set in $\mathbb{R}$.

STEP 1: Define the following equivalence relation on $[0,1]$:

$$x \sim y \Leftrightarrow x - y \text{ is rational}$$

Using $\sim$ we can partition $[0,1]$ into equivalence classes, that is we can write $[0,1]$ as a **disjoint** union

$$[0,1] = \bigcup_{a \in [0,1]} [a]$$

Where $[a] = \{x \mid x \sim a\}$

STEP 2: For every equivalence class $[a]$, choose **exactly** one element x_a from each equivalence class, and let

$$\mathcal{N} = \{x_a\}$$

(This "choosing" step requires the axiom of choice)

STEP 3: $\mathcal{N}$ is not measurable.

By contradiction, suppose $\mathcal{N}$ is measurable.

Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of all the rationals in $[-1,1]$ and consider the translates

$$\mathcal{N}_k =: \mathcal{N} + r_k$$

We claim that $\mathcal{N}_k$ are disjoint and

$$[0,1] \subseteq \bigcup_{k=1}^{\infty} \mathcal{N}_k \subseteq [-1,2]$$

Disjoint: Suppose $\mathcal{N}_k \cap \mathcal{N}_p \neq \emptyset$. Then there are rationals $r_k \neq r_p$ and a and b such that $x_a + r_k = x_b + r_p$ but then $x_a - x_b = r_p - r_k \in \mathbb{Q}$ and hence $x_a \sim x_b$ which contradicts the fact that we chose *exactly* one element from each equivalence class

Inclusions: If $x \in [0,1]$ then $x \sim x_a$ for some a and hence $x - x_a = r_k$ for some k and so $x \in \mathcal{N}_k$ and the second inclusion holds since each $\mathcal{N}_k$ is contained in $[-1,2]$ by construction

STEP 4: Conclusion

If each $\mathcal{N}$ were measurable, then so would $\mathcal{N}_k$ for all k (by translation) and since the union $\bigcup_{k=1}^{\infty} \mathcal{N}_k$ is disjoint, the above would imply:

$$m([0,1]) \leq m\left(\bigcup_{k=1}^{\infty} \mathcal{N}_k\right) \leq \quad m([-1,2])$$

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}_k) \leq 3$$

Since $\mathcal{N}_k$ is a translate of $\mathcal{N}$, we have $m(\mathcal{N}_k) = m(\mathcal{N})$ and hence

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}) \leq 3$$

Hence a contradiction, since neither $m(\mathcal{N}) = 0$ or $m(\mathcal{N}) > 0$ holds $\square$

Note: It can be shown that $m_\star(\mathcal{N}) = c > 0$ for some c. The crucial property that fails is $m_\star(E_1 \cup E_2) \neq m_\star(E_1) + m_\star(E_2)$.

Aside: Surprisingly, without the axiom of choice, every set is measurable! But then without AC it's quite impossible to build all of analysis

4. Measurable Functions

Let's now generalize measurability to functions.

Definition: If E is measurable and $f : E \to \mathbb{R}$, then f is **measurable** if for all $a \in \mathbb{R}$ the set

$$\{f < a\} =: \{x \in E \mid f(x) < a\} \text{ is measurable}$$

This is also sometimes written as $f^{-1}(-\infty, a)$. In probability, measurable functions are called random variables.

Note: This is equivalent to requiring that $\{f \leq a\}$ is measurable for all a because

$$\{f \leq a\} = \bigcap_{k=1}^{\infty} \left\{ f < a + \frac{1}{k} \right\}$$

$$\{f < a\} = \bigcup_{k=1}^{\infty} \left\{ f \leq a - \frac{1}{k} \right\}$$

This is also equivalent requiring $\{f \geq a\} = \{f < a\}^c$ to be measurable

And if f is finite-valued ($f(x) \neq \pm\infty$) this is the same thing as requiring $\{a < f < b\}$ to be measurable. And in fact, by the same argument:

Property 1: If f is finite valued, then f is measurable if and only if $f^{-1}(O)$ is measurable for every open set O

Note: Compare this with continuity, which says $f^{-1}(O)$ is open whenever O is open, so it's a more general version of continuity.

Property 2: If f is measurable and finite-valued and Φ is continuous, then $\Phi \circ f$ is measurable.

Proof: If O is open, then $(\Phi \circ f)^{-1}(O) = f^{-1}\left(\underbrace{\Phi^{-1}(O)}_{\text{open}}\right) = \text{ Measurable}$

The last step follows since f is measurable $\square$

Note: The order matters! If f is measurable and Φ is continuous, then $f \circ \Phi$ might not be measurable!

In the same spirit, let's show that measurability is preserved under familiar operations

Property 3: If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions, then the following are measurable

$$\sup_n f_n(x) \text{ and } \inf f_n(x) \text{ and } \limsup_{n\to\infty} f_n(x) \text{ and } \liminf_{n\to\infty} f_n(x)$$

Proof: Notice $\{\sup_n f_n > a\} = \bigcup_n \{f_n > a\}$

For inf, remember that $\inf(f_n) = -\sup(-f_n)$

Finally, the $\limsup / \liminf$ part follows from the above as well as

$$\limsup_{n\to\infty} f_n(x) = \inf_k \left\{ \sup_{n\geq k} f_n \right\} \text{ and } \liminf_{n\to\infty} f_n(x) = \sup_k \left\{ \inf_{n\geq k} f_n \right\}$$

Property 4: If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions and

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ pointwise , then } f \text{ is measurable}$$

This just follows because in that case $f(x) = \limsup_{n\to\infty} f_n(x)$

5. Appendix: Proof of Proposition 4

Property 4: Closed sets F are measurable

Partial Proof: For this, we'll need two facts[1]

Fact 1: Every open subset of $\mathbb{R}^d$ is the countable union of almost disjoint cubes.

Fact 2: If F is closed and K is compact, then $d(F, K) > 0$

WLOG, assume F is compact, because otherwise consider $F = \bigcup_{k=1}^{\infty} \underbrace{F \cap B(0,k)}_{\text{Compact}}$ and the result would follow because the countable union of measurable sets is measurable.

Let $\epsilon > 0$ be given, then since $m_\star(F) = \inf m_\star(O)$, there is an open set $O \supseteq F$ with $m_\star(O) \leq m_\star(F) + \epsilon$.

[1] For proofs, check out Theorem 1.4 and Lemma 3.1 of Stein and Shakarchi

Since F is closed, the difference $O - F$ is open, so by Fact 1 above, we can write $O - F = \bigcup_{j=1}^{\infty} Q_j$ where the Q_j are almost disjoint cubes.

For fixed N, the finite union $K = \bigcup_{j=1}^{N} Q_j$ is compact, and hence by Fact 2 above, $d(F, K) > 0$.

But since $F \cup K \subseteq O$ we have

$$m_\star(O) \geq m_\star(F \cup K) = m_\star(F) + m_\star(K) = m_\star(F) + \sum_{j=1}^{N} m_\star(Q_j)$$

Hence $\sum_{j=1}^{N} m_\star(Q_j) \leq m_\star(O) - m_\star(F) \leq \epsilon$. Then let $N \to \infty$ to conclude $\sum_{j=1}^{\infty} m_\star(Q_j) \leq \epsilon$ and finally we have $m_\star(O - F) \leq \sum_{j=1}^{\infty} m_\star(Q_j) \leq \epsilon$ $\square$