LECTURE 2: UNIFORM CONVERGENCE

1. The space $C[a,b]$

What makes uniform convergence so powerful is that it allows us to talk about distances between two functions:

Definition: $C[a,b]$ is the set of continuous functions $f : [a,b] \to \mathbb{R}$

Definition: If $f \in C[a,b]$, then

$$\|f\| = \sup\{|f(x)|, x \in [a,b]\}$$

This is sometimes called the sup-norm (or infinity norm) and is sometimes written as $\|f\|_{\infty}$ or $\|f\|_0$

The distance between f and g is nothing other than

$$\|f - g\| = \sup\{|f(x) - g(x)|, x \in [a,b]\}$$

$\|f - g\|$ measures the biggest possible spread between f and g (see picture in lecture)

This distance turns $C[a,b]$ into a **metric space**

Fact: $(C[a,b], d)$ is a metric space, where $d(f,g) = \|f - g\|$

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Metric spaces were useful in Analysis 1 because they allowed us to talk about convergence:

Reminder: $x_n \to x$ (in a metric space) if for all $\epsilon > 0$ there is N such that if $n > N$ then $d(x_n, x) < \epsilon$

And in fact convergence in $C[a, b]$ is none other than uniform convergence!

Fact: $f_n \to f$ in $C[a, b] \Leftrightarrow f_n \to f$ uniformly.

Proof: ($\Rightarrow$) Let $\epsilon > 0$ be given. Then since $f_n \to f$ in $C[a, b]$, there is N such that if $n > N$ then $d(f_n, f) < \epsilon$, that is

$$\sup \{|f_n(x) - f(x)|, x \in [a, b]\} < \epsilon$$

(But if a sup is $< \epsilon$, then all its values are $< \epsilon$)

With the same N, $n > N$ then for all x, we have $|f_n(x) - f(x)| < \epsilon$, so $f_n \to f$ uniformly.

($\Leftarrow$) Similar □

2. Completeness

Not only is this a metric space but it's a *complete* metric space, it has no holes:

Recall: (x_n) is **Cauchy** if for all $\epsilon > 0$ there is N such that if $m, n > N$ then $d(x_n, x_m) < \epsilon$

(That is, the sequence eventually gets closer and closer to each other, just like people gathering in a crowd)

Recall: A metric space is **complete** if every Cauchy sequence converges.

Theorem: $(C[a,b], d)$ is **complete**

Proof:[1]

STEP 1: Let f_n be a Cauchy sequence in $C[a,b]$.

Claim: For every x, $(f_n(x))$ is Cauchy (in $\mathbb{R}$)

Why? Let $\epsilon > 0$ be given, then there is N such that if $m, n > N$ then $d(f_n, f_m) < \epsilon$. With that same N, if $m, n > N$ then

$$|f_n(x) - f_m(x)| \leq \sup \{|f_n(x) - f_m(x)|, x \in [a,b]\} = d(f_n, f_m) < \epsilon$$

STEP 2: Since $(f_n(x))$ is Cauchy in $\mathbb{R}$, it converges. So for every x, it makes sense to define

$$f(x) =: \lim_{n \to \infty} f_n(x)$$

And, by definition, $f_n \to f$ pointwise

STEP 3: **Claim:** $f_n \to f$ uniformly

Why? Let $\epsilon > 0$ be given. Since (f_n) is Cauchy, there is N such that if $m, n > N$ then

$$d(f_n, f_m) < \frac{\epsilon}{2}$$

[1] This proof is taken from Pugh's Real Analysis book, Theorem 3 in Chapter 4

Take that N and let x be given

Since $f_n \to f$ pointwise, we know there is some m (depending on x) large enough such that $|f_m(x) - f(x)| < \frac{\epsilon}{2}$ (think of it as a helper constant)

Then, if $n \geq N$, we get

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Where we have used Cauchiness and our helper constant respectrively.

Finally $f \in C[a, b]$ since the uniform limit of continuous functions is continuous □

Let's isolate this as a separate fact, since we'll use it often:

Fact: (f_n) converges uniformly if and only if (f_n) is uniformly Cauchy, that is for all $\epsilon > 0$ there is N such that for all $m, n > N$ and all x, we have $|f_m(x) - f_n(x)| < \epsilon$

(We have shown the if part, and the other part is a standard $\frac{\epsilon}{2}$ argument)

3. Uniform Convergence and Differentiation

Using those new tools, let's go back to differentiability:

Theorem: (Differentiability)

(1) Suppose f_n is differentiable on $[a, b]$ and $f_n \to f$ uniformly

(2) Moreover, suppose $f_n' \to g$ uniformly for some function g

(3) Then in fact f is differentiable and $f' = g$.

Proof: Now let's do the general case

STEP 1: In general, we need to work with *difference quotients*[2]

Fix some $x \in [a, b]$ and define

$$\phi_n(t) = \begin{cases} \frac{f_n(t)-f_n(x)}{t-x} & \text{if } t \neq x \\ f_n'(x) & \text{if } t = x \end{cases}$$

Claim # 1: Each ϕ_n is continuous

Why? We only need to check that ϕ_n is continuous at x. But by definition of the derivative, we have:

$$\lim_{t \to x} \phi_n(t) = \lim_{t \to x} \frac{f_n(t) - f_n(x)}{t - x} = f_n'(x) = \phi_n(x) \checkmark$$

Claim # 2: $\phi_n \to \phi$ pointwise, where

$$\phi(t) = \begin{cases} \frac{f(t)-f(x)}{t-x} & \text{if } t \neq x \\ g(x) & \text{if } t = x \end{cases}$$

Why? If $t \neq x$, then since $f_n \to f$ pointwise, we get

$$\lim_{n \to \infty} \phi_n(t) = \lim_{n \to \infty} \frac{f_n(t) - f_n(x)}{t - x} = \frac{f(t) - f(x)}{t - x}$$

If $t = x$, then since $f_n' \to g$, we get

$$\lim_{n \to \infty} \phi_n(x) = \lim_{n \to \infty} f_n'(x) = g(x) \checkmark$$

[2] you have dealt with difference quotients before when you proved the Chain Rule

Claim # 3: $\phi_n \to \phi$ uniformly

Once we prove Claim # 3, we're done with the proof, because ϕ_n continuous and $\phi_n \to \phi$ uniformly implies ϕ is continuous, and hence

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \phi(t) = \phi(x) = g(x)$$

Hence $f'(x)$ exists and equals $g(x)$ ✓

STEP 2: Proof of Claim # 3 Since it is difficult to deal with ϕ_n directly, let's use the Cauchy criterion, so consider:

$$\begin{aligned}
\phi_m(t) - \phi_n(t) =& \frac{f_m(t) - f_m(x)}{t - x} - \frac{f_n(t) - f_n(x)}{t - x} \\
=& \frac{(f_m - f_n)(t) - (f_m - f_n)(x)}{t - x} \\
=& \left(f_m - f_n\right)'(c) \\
=& f'_m(c) - f'_n(c)
\end{aligned}$$

(If $t = x$, we get the same result)

Where we used the Mean Value Theorem applied to $f_m - f_n$, where c is some number between x and g

Let $\epsilon > 0$ be given, then since $f'_n \to g$, there is N such that for all $m, n > N$ we have $\|f'_m - f'_n\| < \epsilon$.

With that N, if $m, n > N$ we get for all t,

$$|\phi_m(t) - \phi_n(t)| = |f'_m(c) - f'_n(c)| \leq \epsilon$$

Therefore ϕ_n converges uniformly to some function, which must be ϕ (since ϕ_n already converges pointwise to ϕ) □

4. Series of Functions

The cool thing is that everything we talked about also works series of functions, provided we consider partial sums:

Definition: The series $\sum_{n=0}^{\infty} f_n(x)$ **converges uniformly** if the sequence of partial sums $F_n(x)$ converges uniformly, where

$$F_n(x) = \sum_{k=0}^{n} f_k(x)$$

(Rudin uses s_n, but here I want to emphasize that those are functions)

Example: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly for all x, and this series we call e^x. See next chapter

Luckily, there is a **very** important way of checking if a series converges uniformly. It's kind of like a comparison test for series

Theorem: [Weierstraß M-test]

Suppose for all x and all n,

$$|f_n(x)| \leq M_n$$

Where M_n are constants. If $\sum M_n$ converges, then $\sum f_n$ converges uniformly (and absolutely, that is $\sum |f_n|$ converges)

So instead of checking that a series of functions converges, you just need to check that a series of numbers converges.

Example: Does $\sum \frac{1}{x^2+n^2}$ converge uniformly?

Notice $\frac{1}{x^2+n^2} \leq \frac{1}{n^2} =: M_n$ and since $\sum M_n = \sum \frac{1}{n^2}$ converges, the original series converges.

Proof of Weierstraß Let $\epsilon > 0$ be given. Since $\sum M_n$ converges, by the Cauchy criterion for series, there is N such that if $n \geq m > N$ then

$$\sum_{k=m}^{n} M_k < \epsilon$$

(Cauchy criterion just means that, after a long time, the tail of the series is arbitrarily small)

With the same N, for all x, if $n \geq m > N$ then

$$\left| \sum_{k=m}^{n} f_k(x) \right| \leq \sum_{k=m}^{n} |f_k(x)| \leq \sum_{k=m}^{n} M_k < \epsilon$$

So $\sum f_k$ converges uniformly, again by the Cauchy criterion for series and because N doesn't depend on x $\square$

Just to show you again that series can behave weirdly, consider:

Non-Example: Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$ and let

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

Since $f_n(0) = 0$ we have $f(0)$. But if $x \neq 0$ we have:

$$f(x) = x^2 \sum_{n=0}^{\infty} \left(\frac{1}{1+x^2}\right)^n = x^2 \left(\frac{1}{1-\frac{1}{1+x^2}}\right) = x^2 \left(\frac{1+x^2}{x^2}\right) = 1 + x^2$$

Therefore the series converges to

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 + x^2 & \text{if } x \neq 0 \end{cases}$$

So a convergent series of continuous functions might have a discontinuous sum.

5. Integration and Differentiation of Series

What makes uniformly convergent series so nice is that they can be integrated and differentiated term by term. So a lot of "illegal" operations in calculus and physics are actually legitimate for uniformly convergent series.

Theorem: [Term by Term Integration] If $\sum_n f_n$ converges uniformly and each f_n is integrable, then

$$\int_a^b \sum_{n=0}^{\infty} f_n(x)dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)dx$$

Proof: Consider the partial sum

$$\sum_{k=0}^{n} \int_a^b f_k(x)dx$$

On the one hand, by definition of a series, as $n \to \infty$, the above converges to

$$\sum_{n=0}^{\infty} \int_a^b f_n(x)dx$$

On the other hand, since it's just a finite sum, we have

$$\sum_{k=0}^{n} \int_a^b f_k(x)dx = \int_a^b \sum_{k=0}^{n} f_k(x)dx = \int_a^b F_n(x)dx$$

By definition of a series, $F_n(x)$ converges uniformly to $\sum_{n=0}^{\infty} f_n(x)dx$, so by the integration result from last time, we get

$$\int_a^b F_n \to \int_a^b \sum_{n=0}^{\infty} f_n(x)$$

Comparing the two limits, we get our desired result □

Theorem: [Term by Term Differentiation] If $\sum_n f_n$ and $\sum_n f_n'$ converge uniformly, then

$$\left(\sum_{n=0}^{\infty} f_n\right)' = \sum_{n=0}^{\infty} f_n'(x)$$

The proof is exactly the same as above, but with derivatives.