LECTURE 20: MIDTERM 2 – REVIEW

1. Let's go on a tangent

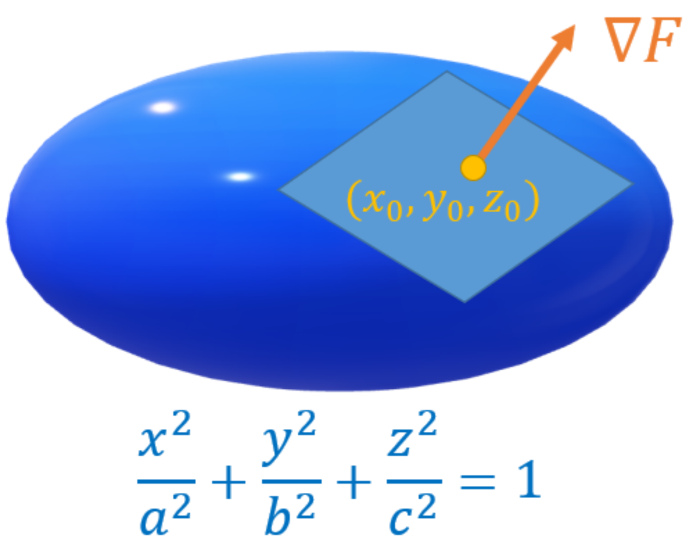
Example 1:

Show that the tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at the point (x_0, y_0, z_0) can be written as

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1$$



Date: Wednesday, October 13, 2021.

(Even though this is not a function, we can still find the tangent plane, see 14.6)

STEP 1:

$$F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

STEP 2: Gradient

$$\nabla F = \langle F_x, F_y, F_z \rangle = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle$$

So at (x_0, y_0, z_0) this becomes:

$$\mathbf{n} = \nabla F(x_0, y_0, z_0) = \left\langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2} \right\rangle$$

STEP 3: Equation

Point: (x_0, y_0, z_0), Normal vector $\mathbf{n}$ (as above), so:

$$\left(\frac{2x_0}{a^2}\right)(x - x_0) + \left(\frac{2y_0}{c^2}\right)(y - y_0) + \left(\frac{2z_0}{z^2}\right)(z - y_0) = 0$$

STEP 4: Clean-up And we just need to clean it up to make it look like the equation in the problem:

$$\begin{aligned}
\frac{x_0 x}{a^2} - \frac{(x_0)^2}{a^2} + \frac{y_0 y}{b^2} - \frac{(y_0)^2}{b^2} + \frac{z_0 z}{c^2} - \frac{(z_0)^2}{c^2} &= 0 \\
\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} &= \frac{(x_0)^2}{a^2} + \frac{(y_0)^2}{b^2} + \frac{(z_0)^2}{c^2} \\
\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} &= 1 \checkmark
\end{aligned}$$

In the last part, we used that (x_0, y_0, z_0) is on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and therefore satisfies the equation of the ellipsoid.

2. Use The Chen Lu

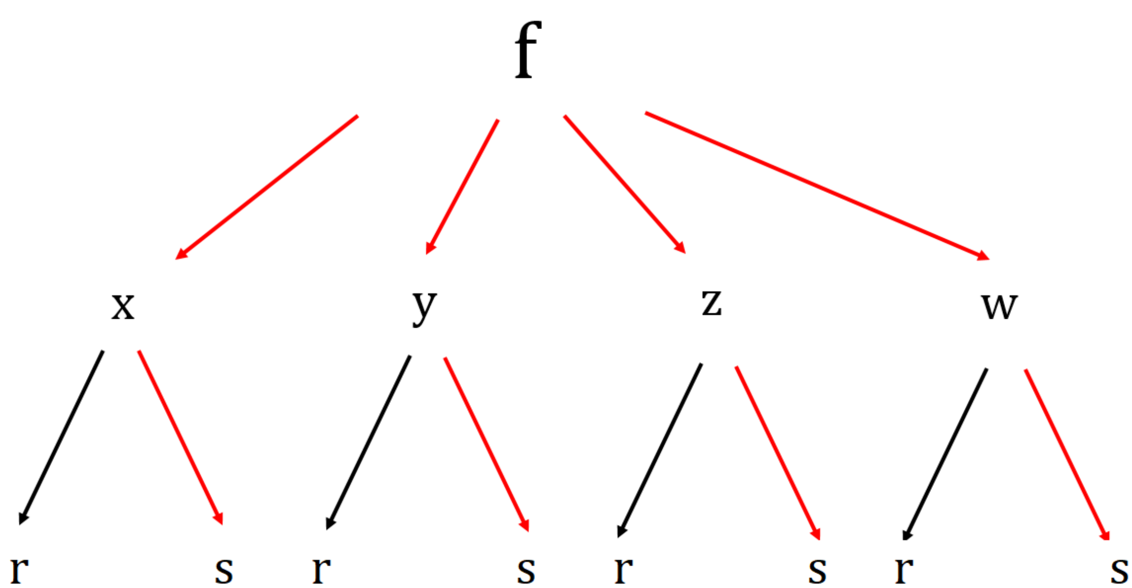
Of course no Peyam review would be complete without the Chen Lu, so let's use the Chen Lu!

Example 2:

Find $\frac{\partial f}{\partial s}$ where $f = f(x, y, z, w)$ and:

$$\begin{cases} x = & s^2 \\ y = & r^2 s^2 \\ z = & r^3 \\ w = & re^s \end{cases}$$

It's all about the diagram!



$$\begin{aligned}\frac{\partial f}{\partial s} &= \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial s}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial s}\right) + \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial z}{\partial s}\right) + \left(\frac{\partial f}{\partial w}\right)\left(\frac{\partial w}{\partial s}\right) \\ &= \left(\frac{\partial f}{\partial x}\right)\left(s^2\right)_s + \left(\frac{\partial f}{\partial y}\right)\left(r^2 s^2\right)_s + \left(\frac{\partial f}{\partial z}\right)\left(r^3\right)_s + \left(\frac{\partial f}{\partial w}\right)\left(re^s\right)_s \\ &= \left(\frac{\partial f}{\partial x}\right)(2s) + \left(\frac{\partial f}{\partial y}\right)\left(2r^2 s\right) + \left(\frac{\partial f}{\partial w}\right)\left(re^s\right)\end{aligned}$$

3. Maximize your happiness

Let's maximize our happiness with a max/min problem!

Example 3:

Find the local max/min/saddle point (values) of the following:

$$f(x,y) = x^3 - 3xy + y^3$$

STEP 1: Critical Points:

$$\begin{aligned} f_x &= \left(x^3 - 3xy + y^3\right)_x = 3x^2 - 3y = 0 \Rightarrow 3y = 3x^2 \Rightarrow y = x^2 \\ f_y &= \left(x^3 - 3xy + y^3\right)_y = -3x + 3y^2 = 0 \Rightarrow 3x = 3y^2 \Rightarrow x = y^2 \end{aligned}$$

Now take $x = y^2$ and plug in $y = x^2$ to get:

$$x = y^2 \Rightarrow x = \left(x^2\right)^2 \Rightarrow x = x^4$$

So $x - x^4 = 0$, so $x(1-x^3) = 0$ so $x = 0$ or $x^3 = 1$, that is $x = 0$ or $x = 1$.

Case 1: $x = 0$, then $y = x^2 = 0^2 = 0$ which gives $(0,0)$

Case 2: $x = 1$, then $y = x^2 = 1^2 = 1$ which gives $(1,1)$

Hence there are two critical points: $(0,0),(1,1)$

STEP 2: Second Derivatives:

Recall: $f_x = 3x^2 - 3y,\ f_y = -3x + 3y^2$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix}$$

$$D(0,0) = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = (0)(0) - (-3)(-3) = -9 < 0$$

Hence $(0,0)$ is a **saddle**

$$D(1,1) = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 36 - 9 = 27 > 0$$

And $f_{xx}(1,1) = 6 > 0$, hence f has a local min at $(1,1)$

STEP 3: Conclusion:

Saddle point at $(0,0)$ and $f(0,0) = 0$

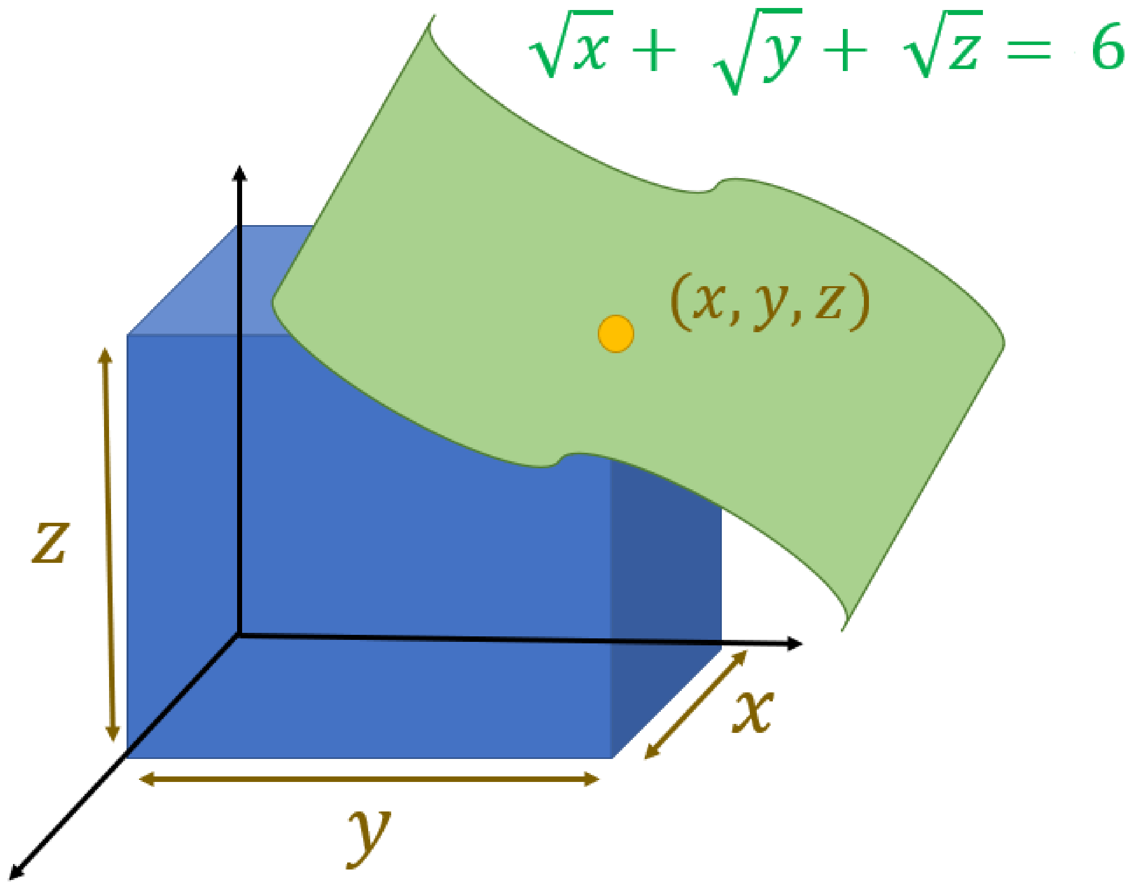
Local min at $(1,1)$ and $f(1,1) = 1 - 3 + 1 = -1$

4. Derivative in a Box

Example 4:

Find the volume of the largest box in the first octant with three faces in the coordinate planes and one vertex on the surface

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = 6$$



STEP 1: Find f and g

$$V = xyz \Rightarrow f(x,y,z) = xyz$$

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = 6 \Rightarrow g(x,y,z) = \sqrt{x} + \sqrt{y} + \sqrt{z} - 6$$

STEP 2: Lagrange Equation: $\nabla f = \lambda \nabla g$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ \text{Constraint} \end{cases} \Rightarrow \begin{cases} yz = \dfrac{\lambda}{2\sqrt{x}} \\ xz = \dfrac{\lambda}{2\sqrt{y}} \\ xy = \dfrac{\lambda}{2\sqrt{z}} \\ \sqrt{x} + \sqrt{y} + \sqrt{z} = 6 \end{cases} \Rightarrow \begin{cases} \lambda = 2\left(\sqrt{x}\right) yz \\ \lambda = 2\left(\sqrt{y}\right) xz \\ \lambda = 2\left(\sqrt{z}\right) xy \\ \sqrt{x} + \sqrt{y} + \sqrt{z} = 6 \end{cases}$$

STEP 3: Equating the first two equations we get

$$\begin{aligned} 2\left(\sqrt{x}\right) yz =& 2\left(\sqrt{y}\right) xz \\ \left(\sqrt{x}\right) y =& \left(\sqrt{y}\right) x \\ \frac{y}{\sqrt{y}} =& \frac{x}{\sqrt{x}} \\ \sqrt{y} =& \sqrt{x} \\ y =& x \end{aligned}$$

Similarly, equating the second equation with the third we get

$$2\left(\sqrt{y}\right) xz = 2\left(\sqrt{z}\right) xy \Rightarrow y = z$$

Therefore $x = y = z$ and the optimal box is a cube

STEP 4: Constraint

$$\begin{aligned} \sqrt{x} + \sqrt{y} + \sqrt{z} &= 6 \\ \sqrt{x} + \sqrt{x} + \sqrt{x} &= 6 \\ 3\sqrt{x} &= 6 \\ \sqrt{x} &= 2 \\ x &= 4 \end{aligned}$$

STEP 5: **Answer:** $x = y = z = 4$ and so the largest volume is

$$V = xyz = 4 \times 4 \times 4 = 64$$

5. Differential, but not different

Example 5:

Use **differentials** to find an approximate value of

$$(2.96)^2 + (2.96)(0.95) + (0.95)^2$$

STEP 1: **Prep work:**

$$\begin{aligned} f(x,y) &= x^2 + xy + y^2 \\ \text{Point } &= (3,1) \\ dx &= 2.96 - 3 = -0.04 \\ dy &= 0.95 - 1 = -0.05 \end{aligned}$$

STEP 2:

$$\begin{aligned} dz &= \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy \\ &= (2x+y)(dx) + (x+2y)\,dy \\ &= (2(3)+1)(-0.04) + (3+2(1))(-0.05) \\ &= 7(-0.04) + 5(-0.05) \\ &= -0.28 - 0.25 \\ &= -0.53 \end{aligned}$$

STEP 3:

$$
\begin{aligned}
\Delta z &\approx dz \\
f(2.96, 0.95) - f(3, 1) &\approx -0.53 \\
f(2.96, 0.95) &\approx f(3, 1) - 0.53
\end{aligned}
$$

But $f(3, 1) = 3^2 + (3)(1) + 1^2 = 9 + 3 + 1 = 13$

$$f(2.96, 0.95) \approx 13 - 0.53 = 12.47$$