LECTURE 20: THE LEBESGUE INTEGRAL (I)

1. Measurable Functions

Definition: f is **measurable** if for all $a \in \mathbb{R}$

$$\{f < a\} = \{x \mid f(x) < a\} \text{ is measurable}$$

Property 5: If f and g are measurable, then so is f^k for any k and (if f and g are finite-valued), so are $f + g$ and fg

Proof: (1) If k is odd, notice $\left\{f^k < a\right\} = \left\{f < a^{\frac{1}{k}}\right\}$ and similar if even

(2) $f + g$ is measurable because

$$\{f + g < a\} = \bigcup_{r \in \mathbb{Q}} \{f < r\} \cap \{g < a - r\}$$

Finally, for fg use the above and

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right] \quad \square$$

Definition: $f = g$ **almost everywhere** if $f(x) = g(x)$ for every x except for a set of measure 0. That is, $m\{x \mid f(x) \neq g(x)\} = 0$

In particular, $\{f < a\}$ and $\{g < a\}$ differ by a set of measure 0, so if one is measurable then the other one is as well. To summarize:

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Property 6: If f is measurable and $f(x) = g(x)$ almost everywhere, then g is measurable.

All the properties discussed work almost everywhere. For example if $\{f_n\}$ is measurable and $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere, then f is measurable

2. Approximations by Simple Functions

The building block of Lebesgue integration is a simple function:

Definition: The **characteristic function** of a set E is:

$$1_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

(This is also called the indicator function)

Definition: A **simple function** is a finite sum

$$f = \sum_{k=1}^{N} a_k \, 1_{E_k}$$

Each E_k is a measurable set of finite measure, and a_k are constants

(Compare this with the Riemann sum, where you approximate f with rectangles, here you approximate f with measurable sets)

What makes the Lebesgue integral work is that you can approximate measurable functions with simple functions:

Theorem: If $f \geq 0$ then there is an increasing sequence of simple functions $\{\phi_k\}_{k=1}^{\infty}$ that converges pointwise to f, that is

$$\phi_k(x) \leq \phi_{k+1}(x) \quad \text{and} \quad \lim_{k\to\infty} \phi_k(x) = f(x) \text{ for all } x$$

Proof:

STEP 1: First, let's begin with a truncation.

For $k \geq 1$, let Q_k be the cube centered at 0 and sidelength k[1], and let

$$F_k(x) = \begin{cases} f(x) & \text{if } x \in Q_k \text{ and } f(x) \leq k \\ k & \text{if } x \in Q_k \text{ and } f(x) > k \\ 0 & \text{otherwise} \end{cases}$$

This says cut off f at k if f becomes too large.

Then for all x, $\lim_{k\to\infty} F_k(x) = f(x)$ and F_k is increasing with k

This F_k kind of does the job, except it's not a simple function.

STEP 2: Now let's partition the range $[0, k]$ of F_k.

$$\text{Let } E_{k,j} = \left\{ x \in Q_k \mid \frac{j}{k} < F_k(x) \leq \frac{j+1}{k} \right\} \text{ for } 0 \leq j \leq k^2 - 1$$

And form the lower sum

$$\phi_k(x) = \sum_{j=0}^{k^2-1} \left(\frac{j}{k}\right) 1_{E_{k,j}}(x)$$

[1] in $\mathbb{R}$ that's just the interval $\left(-\frac{k}{2}, \frac{k}{2}\right)$

Then ϕ_k is simple and $0 \leq F_k(x) - \phi_k(x) \leq \frac{j+1}{k} - \frac{j}{k} = \frac{1}{k}$ (for some j)

From this it follows that $\lim_{k\to\infty} F_k - \phi_k = 0$ pointwise and since $F_k \to f$ pointwise, we get $\phi_k \to f$ pointwise. Finally, ϕ_k is increasing with k because F_k is (the lower sums just become bigger)

A similar statement is true if f isn't non-negative any more, provided you use absolute values:

Theorem: For general measurable f, there is a sequence of simple functions $\{\phi_k\}_{k=1}^{\infty}$ such that

$$|\phi_k(x)| \leq |\phi_{k+1}(x)| \text{ and } \lim_{k\to\infty} \phi_k(x) = f(x) \text{ for all } x$$

In particular $|\phi_k(x)| \leq |f(x)|$ for all x and k

Proof: Classical procedure when dealing with functions of mixed sign:

Notice $f = f^+ - f^-$ where $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$

Since $f^\pm \geq 0$ the previous result gives us two increasing sequences of non-negative simple functions $\left\{\phi_k^+\right\}$ and $\left\{\phi_k^-\right\}$ that converge pointwise to $f^\pm$. Then if

$$\phi_k(x) =: \phi_k^+(x) - \phi_k^-(x)$$

Then $\phi_k(x)$ converges to $f^+(x) - f^-(x) = f(x)$ for all x and moreover

$$|\phi_k(x)| = \phi_k^+(x) + \phi_k^-(x)$$

Since each $\phi_k^\pm$ is increasing, it follows that $|\phi_k|$ is increasing □

3. Egorov's and Lusin's Theorem

Finally, let's state two more facts about functions, one is that "every convergent sequence is almost uniformly convergent" (Egorov), the other is that "every function is almost continuous" (Lusin).

Egorov's Theorem: If $m(E) < \infty$ and $f_k : E \to \mathbb{R}$ with $f_k \to f$ a.e. on E. Then, if $\epsilon > 0$ is given, there is a closed subset $A_\epsilon \subseteq E$ such that $f_k \to f$ uniformly on A_ϵ and $m(E - A_\epsilon) \leq \epsilon$

So f_k converges uniformly on f on "almost all" of E. Carefully note that this is not the same as $f_k \to f$ almost everywhere!

The proof (see Stein and Shakarchi Theorem 4.4) considers the sets $|f_j(x) - f(x)| < \frac{1}{n}$ in a clever way, and a result from the homework about increasing sets

Lusin's Theorem: If $m(E) < \infty$ and f is finite-valued on E then for every $\epsilon > 0$ there is a closed set $F_\epsilon \subseteq E$ such that $f\,|_{F_\epsilon}$ is continuous and $m(E - F_\epsilon) \leq \epsilon$

The proof (see Stein and Shakarchi Theorem 4.5) uses approximation of f with step functions (= simple functions whose base are rectangles), Egorov's theorem, and the fact that the uniform limit of continuous functions is continuous

With our knowledge of measurable sets and measurable functions, we are now ready to tackle the Lebesgue integral. It is defined in stages, and in each stage we will discover the relevant theorems.

4. Level 1: Simple Functions

Suppose ϕ is a simple function, that is

$$\phi(x) = \sum_{k=1}^{N} a_k \, 1_{E_k}(x)$$

WLOG, assume that each a_k is nonzero and distinct, and each E_k is disjoint (otherwise just group sets with common value together)

Definition: The Lebesgue Integral of $\phi = \sum_{k=1}^{N} a_k \, 1_{E_k}$ is

$$\int_{\mathbb{R}^d} \phi(x) dx = \sum_{k=1}^{N} a_k m(E_k)$$

$$\int_E \phi(x) dx = \int_{\mathbb{R}^d} \phi(x) 1_E(x) dx$$

In the following, ϕ and ψ are simple

Immediate Facts:

(1) (Independence of representation) If $\phi = \sum_{k=1}^{N} a_k \, 1_{E_k}$ is any representation of ϕ, then

$$\int \phi = \sum_{k=1}^{N} a_k m(E_k)$$

(2) Linearity

$$\int a\phi + b\psi = a \int \phi + b \int \psi$$

(3) Additivity: If E and F are disjoint, then

$$\int_{E \cup F} \phi = \int_E \phi + \int_F \phi$$

(4) Monotonicity: If $\phi \leq \psi$ then

$$\int \phi \leq \int \psi$$

(5) Triangle Inequality: $|\phi|$ is simple and

$$\left| \int \phi \right| \leq \int |\phi|$$

(1) The proof is a bit involved and will be skipped. The idea is to first assume that the E_k are disjoint but a_k not distinct, and then just group all the E_k corresponding to common value. If the E_k are not disjoint, then we can just partition the sets E_k until they become disjoint.

(2) follows from the definition

(3) follows because if E and F are disjoint then $1_{E \cup F} = 1_E + 1_F$

(4) enough to show that if $\eta \geq 0$ then $\int \eta \geq 0$, which follows since the coefficients in η are non-negative, and apply this to $\eta = \psi - \phi$.

(5), notice that we have

$$|\phi| = \sum_{k=1}^{N} |a_k| \ 1_{E_k}(x) \text{ and so}$$

$$\left| \int \phi \right| = \left| \sum_{k=1}^{N} a_k m(E_k) \right| \leq \sum_{k=1}^{N} |a_k| \, m(E_k) = \int |\phi|$$

Note: If $f = g$ almost everywhere, then $\int f = \int g$

5. Level 2: Bounded Functions with finite support

Definition: The **support** of f is

$$\text{supp}(f) = \{x \mid f(x) \neq 0\}$$

(Sometimes it's defined as the closure of the above, but the distinction is not important here)

Definition: f is **supported** on E if $f(x) = 0$ whenever $x \notin E$

In this second level, we're interested in bounded functions such that $m(\text{supp}(f)) < \infty$

For this, we will need our Approximation Lemma from before that says there is a sequence of simple functions converging pointwise to f

Definition:

$$\int f(x)dx =: \lim_{k\to\infty} \int \phi_k(x)dx$$

Where $\{\phi_k\}$ is any sequence of bounded simple functions with the same support as f such that $\phi_k \to f$ pointwise.

This raises two important questions: Does the limit exist? And is it independent of the limiting sequence ϕ_k? The answer is contained in the following theorem:

Theorem: With $\{\phi_k\}$ as above, we have

(1) $\lim_{k\to\infty} \int \phi_k$ exists

(2) If $f = 0$ a.e. then $\lim_{k\to\infty} \int \phi_k = 0$

Proof: Let $E = \text{supp}(f)$. This theorem would be obvious *if* $\phi_k \to f$ uniformly on E, but luckily we have Egorov's theorem which says that this is "almost" true.

More precisely, since $m(E) < \infty$ by Egorov there is a closed subset A_ϵ of E such that such that $\phi_n \to f$ uniformly on A_ϵ and $m(E - A_\epsilon) < \epsilon$.

Hence if $I_n =: \int \phi_n$ we get:

$$\begin{aligned} |I_n - I_m| &= \left| \int_E \phi_n - \phi_m dx \right| \\ &\leq \int_E |\phi_n(x) - \phi_m(x)|\, dx \\ &= \left(\int_{A_\epsilon} + \int_{E-A_\epsilon} \right) |\phi_n(x) - \phi_m(x)|\, dx \end{aligned}$$

For the second integral, since each ϕ_n is bounded by M, we get

$$\int_{E-A_\epsilon} |\phi_n(x) - \phi_m(x)|\, dx \leq \int_{E-A_\epsilon} 2M dx = 2Mm(E - A_\epsilon) = 2M\epsilon$$

For the first integral, since ϕ_n converges uniformly on A_ϵ there is N such that if $m, n > N$ then $|\phi_n(x) - \phi_m(x)| < \epsilon$ for all x and so

$$\int_{A_\epsilon} |\phi_n(x) - \phi_m(x)|\, dx \leq \epsilon \int_{A_\epsilon} 1 dx = \epsilon m(A_\epsilon) \leq \epsilon m(E)$$

And therefore, with N as above, if $m, n > N$ we get

$$|I_n - I_m| \leq 2M\epsilon + m(E)\epsilon = (2M + m(E))\epsilon$$

Therefore the sequence $\{I_n\}$ is Cauchy in $\mathbb{R}$ and hence converges, which proves (1).

For (2), if $f = 0$ you can repeat the argument above to show $|I_n| \leq m(E)\epsilon + M\epsilon$ which gives $\lim_{n\to\infty} I_n = 0$ $\square$

It follows that integral is independent of the sequence $\{\phi_k\}$ chosen, because if ϕ_k and ψ_k are two such sequences converging to f, then $\eta_k =: \phi_k - \psi_k$ is also of the same type and converges to $f - f = 0$ a.e. and so by (2) we have

$$\lim_{k\to\infty} \int \eta_k = 0 \Rightarrow \lim_{k\to\infty} \int \phi_k - \psi_k = 0 \Rightarrow \lim_{k\to\infty} \int \phi_k = \lim_{k\to\infty} \int \psi_k \checkmark$$