LECTURE 21: LIMITS (II), DERIVATIVES

1. Properties of limits

Just like for continuity, limits enjoy almost identical properties:

Algebra Facts:

Suppose $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, then:

(1) $\lim_{x\to a} f(x) + g(x) = L_1 + L_2$

(2) $\lim_{x\to a} f(x)g(x) = L_1 L_2$

(3) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$ (provided $L_2 \neq 0$)

There is also a "Chen Lu" (Chain Rule) fact for limits:

Chen Lu Fact:

If $\lim_{x\to a} f(x) = L$ and g continuous at L, then

$$\lim_{x\to a} g(f(x)) = g(L)$$

Proof: Suppose $x_n \to a$, then $f(x_n) \to L$ because $\lim_{x\to a} f(x) = L$. And because g is continuous at L, we get $g(f(x_n)) \to g(L)$ $\square$

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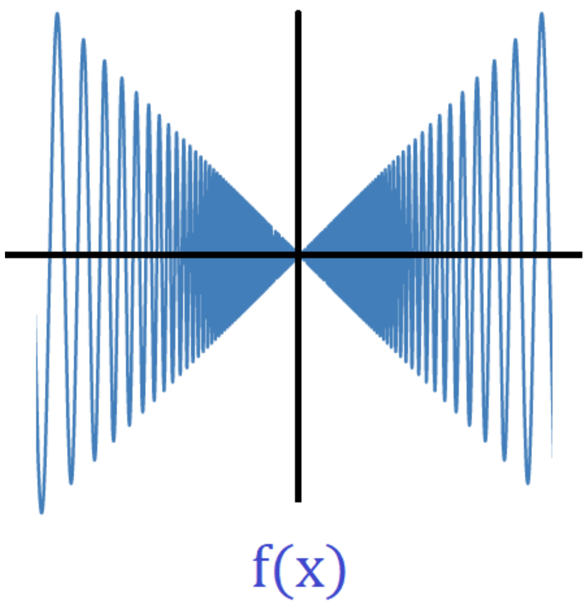
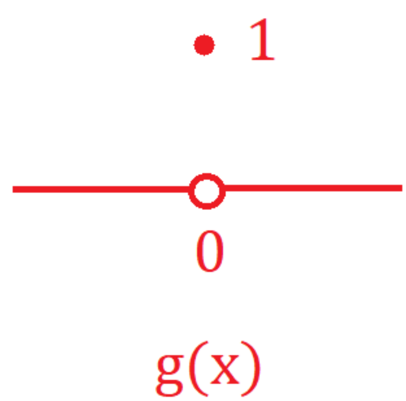
Note: It's important that g be continuous at L, otherwise the result above is wrong:

Non-example 1

Let $f(x) = x\sin\left(\frac{\pi}{x}\right)$ and g be defined as:

$$g(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$





Then $\lim_{x\to 0} f(x) = 0$ (by the squeeze theorem), so the above theorem would say that $\lim_{x\to 0} g(f(x)) = g(0) = 1$

But $\lim_{x\to 0} g(f(x))$ doesn't even exist: if $x_n = \frac{2}{n} \to 0$, then

$$f(x_n) = \frac{2}{n}\sin\left(\frac{\pi}{\frac{2}{n}}\right) = \frac{2}{n}\sin\left(\frac{\pi n}{2}\right)$$

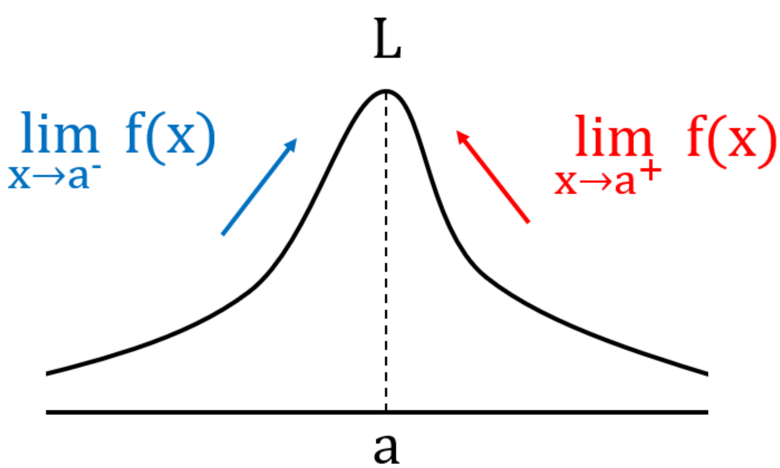
$$\text{Then: } f(x_n) = \begin{cases} 0 & \text{(if } n \text{ is even)} \\ \pm\frac{2}{n} & \text{(If } n \text{ is odd)} \end{cases}$$

So for even n, we have $g(f(x_n)) = g(0) = 1$ and for odd n, we have $g(f(x_n)) = g\left(\pm\frac{2}{n}\right) = 0$. Hence $g(f(x_n))$ is the sequence $(0, 1, 0, 1, 0, 1, \cdots)$ which doesn't converge and certainly cannot converge to $g(0) = 1$

Finally, let's prove a fact relating one-sided limits and two-sided limits that's used throughout calculus:

Two Sided Fact:

$$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$



Proof: Here assume L is finite. The proofs for $L = \pm\infty$ are similar

($\Rightarrow$) Let $\epsilon > 0$ be given, then there is $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$

But, with the same $\delta > 0$, if $0 < x - a < \delta$, then $x - a \leq |x - a| < \delta$ so $|f(x) - L| < \epsilon$ and so $\lim_{x \to a^+} f(x) = L$. Similar with $x \to a^-$.

($\Leftarrow$) Let $\epsilon > 0$ be given

Since $\lim_{x \to a^+} f(x) = L$, there is $\delta_1 > 0$ such that if $0 < x - a < \delta_1$ then $|f(x) - L| < \epsilon$

Since $\lim_{x \to a^-} f(x) = L$, there is $\delta_2 > 0$ such that if $0 < -(x - a) < \delta_2$ then $|f(x) - L| < \epsilon$

Let $\delta = \min\{\delta_1, \delta_2\}$, then if x is such that $0 < |x - a| < \delta$, then:

Case 1: If $x > a$, then $|x - a| < \delta \Rightarrow x - a < \delta < \delta_1$, and so by definition of δ_1, we have $|f(x) - L| < \epsilon$ ✓

Case 2: If $x < a$, then $|x - a| < \delta \Rightarrow -(x - a) < \delta < \delta_2$ and so by definition of δ_2 we have $|f(x) - L| < \epsilon$ ✓

In either case, we get $|f(x) - L| < \epsilon$ $\square$

Note: This would have been terribly hard to prove using sequences, since we wouldn't be able to control whether x_n is $> a$ or $< a$

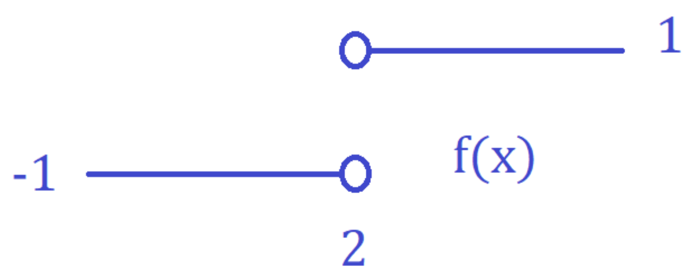
Example 2:

$$f(x) = \frac{|x-2|}{x-2}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x-2}{x-2} = \lim_{x\to 2^+} 1 = 1$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{-(x-2)}{x-2} = \lim_{x\to 2^-} -1 = -1$$

Therefore, by the above fact, $\lim_{x\to 2} f(x)$ doesn't exist.



2. Derivatives

The concept of limits naturally leads to derivatives:

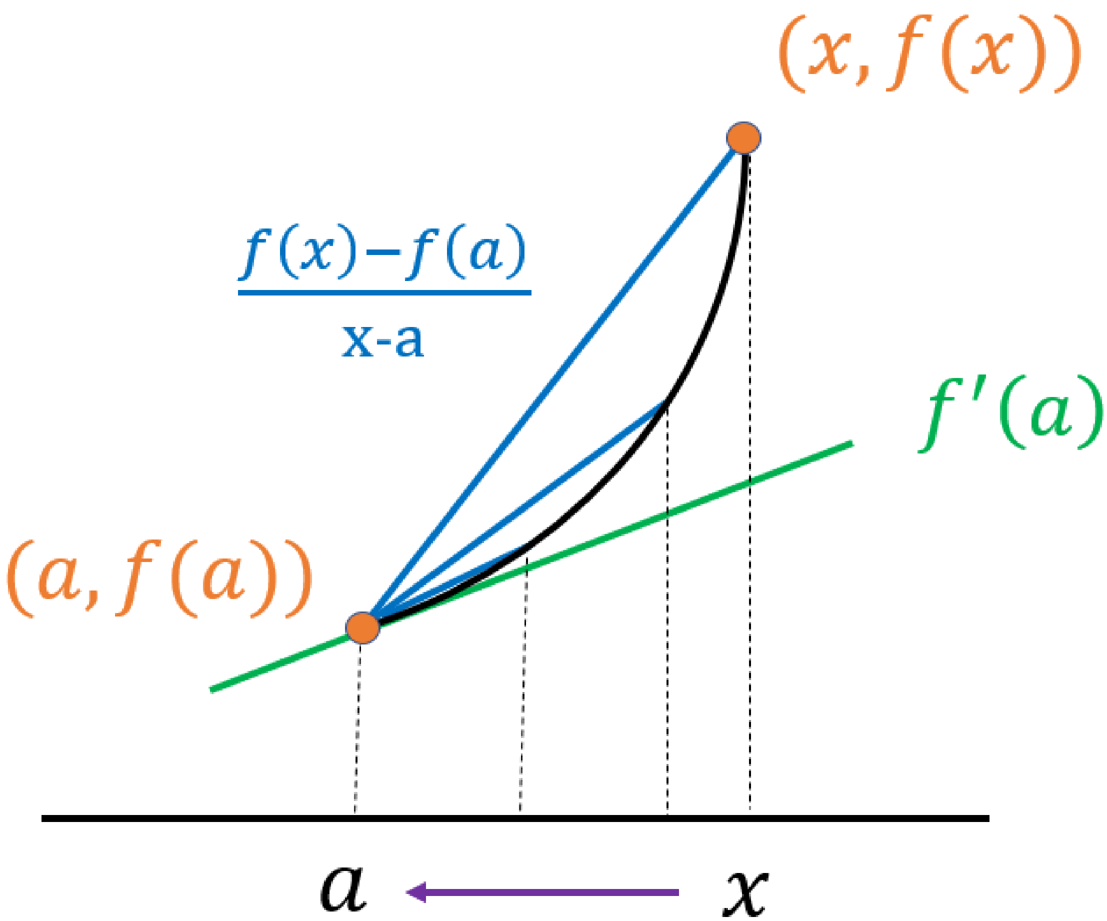
Definition:

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Definition:

We say f is **differentiable** at a provided that the above limit exists and is finite



Interpretation: $\frac{f(x)-f(a)}{x-a}$ is the slope of the secant line connecting $(a, f(a))$ and $(x, f(x))$, while $f'(a)$ is the slope of the tangent line to f at a. The above limit is saying that the slope of the tangent line is the limit of slopes of secant lines as x goes to a

Example 3:

Let $f(x) = x^n$, then:

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{x^n - a^n}{x-a} \\
&= \lim_{x\to a} \frac{\cancel{(x-a)}\left(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}\right)}{\cancel{x-a}} \\
&= \lim_{x\to a} x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1} \\
&= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \cdots + aa^{n-2} + a^{n-1} \\
&= \underbrace{a^{n-1} + a^{n-1} + \cdots + a^{n-1}}_{n \text{ times}} \\
&= na^{n-1}
\end{aligned}$$

Hence $f'(a) = na^{n-1}$, that is $(x^n)' = nx^{n-1}$

Non-Example 4:

Let $f(x) = |x|$, then f is not differentiable at 0 because

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{|x|}{x}$$

But this limit doesn't exist because the left-hand-side and right-hand-side limits are not equal (see Example above)

Theorem:

If f is differentiable at a, then f is continuous at a

Proof: We don't even need $\epsilon - \delta$ for this!

All we need to show is that $\lim_{x \to a} f(x) = f(a)$, but:

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} f(x) - f(a) + f(a) \\
&= \lim_{x \to a} \underbrace{\left(\frac{f(x) - f(a)}{x - a} \right)}_{\to f'(a)} \underbrace{(x - a)}_{\to 0} + f(a) \\
&= f'(a) \times 0 + f(a) \\
&= f(a) \checkmark
\end{aligned}$$

Hence $\lim_{x \to a} f(x) = f(a)$ and f is continuous at a □

3. Properties of Derivatives

Just like limits, derivatives enjoy some special rules (or "Lu"s)

Properties:

If f and g are differentiable at a, then so are $f + g$ and cf for any constant c, and

$$\begin{cases} (f+g)'(a) = f'(a) + g'(a) \\ (cf)'(a) = cf'(a) \end{cases}$$

Proof: Just follows from writing $(f + g)'$ and $(cf)'$ as limits

Video: Product Rule Proof

Product Rule (Prada Lu)

If f and g are differentiable at a, then so is fg and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a)$$

Proof: Similar to what we've done so far with products, by adding and subtracting a term

$$\begin{aligned}
\lim_{x\to a}\frac{(fg)(x)-(fg)(a)}{x-a} &= \lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a}\\
&= \lim_{x\to a}\frac{f(x)g(x)-f(x)g(a)+f(x)g(a)-f(a)g(a)}{x-a}\\
&= \lim_{x\to a}\frac{f(x)g(x)-f(x)g(a)}{x-a}+\lim_{x\to a}\frac{f(x)g(a)-f(a)g(a)}{x-a}\\
&= \lim_{x\to a}\underbrace{f(x)}_{\to f(a)}\underbrace{\left(\frac{g(x)-g(a)}{x-a}\right)}_{\to g'(a)}+\lim_{x\to a}\underbrace{\left(\frac{f(x)-f(a)}{x-a}\right)}_{\to f'(a)}g(a)\\
&= f(a)g'(a)+f'(a)g(a)\\
&= f'(a)g(a)+f(a)g'(a)
\end{aligned}$$

Video: Quotient Rule Proof

Quotient Rule (Koshen Lu)

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$$

Proof: You *could* do it by writing the derivative as a limit (like in the video above), but here we'll do something even cooler! Let's use the product rule to prove the quotient rule:

Let $h(x) = \frac{f(x)}{g(x)}$, then $f(x) = h(x)g(x)$, now differentiating both sides, we get:

$$\begin{aligned} f'(x) &= (h(x)g(x))' \\ f'(x) &= h'(x)g(x) + h(x)g'(x) \\ h'(x)g(x) &= f'(x) - h(x)g'(x) \\ h'(x) &= \frac{f'(x) - h(x)g'(x)}{g(x)} \end{aligned}$$

$$\begin{aligned} \left(\frac{f}{g}\right)'(x) &= \frac{f'(x) - \left(\frac{f(x)}{g(x)}\right) g'(x)}{g(x)} \\ &= \frac{\frac{f'(x)g(x) - f(x)g'(x)}{g(x)}}{g(x)} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \checkmark \end{aligned}$$

Example 5:

Let $f(x) = x^{-n} = \frac{1}{x^n}$, then using the quotient rule you can get $(x^{-n})' = -nx^{-(n+1)}$

4. Use the Chen Lu!

Video: Proof of the Chen Lu

Last but not least, the most powerful Lu of them all:

Chain Rule (Chen Lu)

If f is differentiable at x and g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x and

$$(g \circ f)'(x) = g'(f(x))f'(x)$$

Example 6:

$$\left(\left(x^4 + 13x\right)^7\right)' = 7\left(x^4 + 13x\right)^6\left(4x^3 + 13\right)$$

Wrong Proof: I will first give a wrong proof, and then we'll see how to adapt this to get the correct proof:

$$\begin{aligned}(g \circ f)'(x) &= (g(f(x)))' \\ &= \lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{h} \\ &= \lim_{h \to 0} \left(\frac{g(f(x+h)) - g(f(x))}{f(x+h) - f(x)}\right)\left(\frac{f(x+h) - f(x)}{h}\right) \\ &= g'(f(x))\left(f'(x)\right) \quad \square\end{aligned}$$

What went wrong here? The proof is correct *except* we *could* have $f(x+h) - f(x) = 0$ for small h, in which case we would be dividing by 0, which is a big no-no

It turns out not all is lost, because we *do* have the following observation:

Important Observation: Since $\lim_{h \to 0} \frac{g(x+h)-g(x)}{h} = g'(x)$, we have $\frac{g(x+h)-g(x)}{h} \approx g'(x)$ for h close to 0, and more formally we write this as:

Fact:

$$\frac{g(x+h)-g(x)}{h} = g'(x) + O(h)$$

Where $O(h)$ is a function with the property that $O(h) \to 0$ as $h \to 0$

In fact, solving this, we get $O(h) = \frac{g(x+h)-g(x)}{h} - g'(x)$, which goes to 0 as $h \to 0$, by *definition* of $g'(x)$

Note: This is sometimes called a first-order or linear approximation, since this implies that $g(x+h) = g(x) + hg'(x) + hO(h)$ (like a Taylor series)

Example 7:

If $g(x) = x^3$, then:

$$\begin{aligned}
\frac{g(x+h)-g(x)}{h} &= \frac{(x+h)^3 - x^3}{h} \\
&= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \underbrace{3x^2}_{g'(x)} + 3xh + h^2 \\
&= g'(x) + O(h)
\end{aligned}$$

Where $O(h) = 3xh + h^2 \to 0$ as $h \to 0$