LECTURE 21: THE LEBESGUE INTEGRAL (II)

1. Recap: Level 1 and 2

LEVEL 1: (f simple) The Lebesgue Integral of $\phi = \sum_{k=1}^{N} a_k \, 1_{E_k}$ is

$$\int_{\mathbb{R}^d} \phi(x)dx = \sum_{k=1}^{N} a_k m(E_k)$$

LEVEL 2: (f bounded and finite support)

$$\int f(x)dx = \lim_{k\to\infty} \int \phi_k(x)dx$$

Where $\{\phi_k\}$ is any sequence of bounded simple functions (with same support as f) such that $\phi_k \to f$ pointwise

Last time, we've seen that this limit indeed exists and is independent of the sequence ϕ_k used.

Moreover, you can check that the same properties (linearity, monotonicity, triangle inequality) still hold.

2. Bounded Convergence Theorem

Question: If $f_n \to f$ pointwise, do we have $\int f_n \to \int f$?

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In general, the answer is **NO**:

Non-Example:

$$\text{Let } f_n(x) = \begin{cases} n & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

Then $f_n \to 0$ pointwise but $\int f_n = 1$ for all n so $\int f_n \nrightarrow \int 0$

The Bounded Convergence Theorem says **YES** if all the f_n are bounded:

Bounded Convergence Theorem: If $\{f_n\}$ is a sequence of functions bounded by M (and supported on E with $m(E) < \infty$) and $f_n \to f$ a.e., then

$$\lim_{n \to \infty} \int |f_n - f| = 0$$

Note: In particular implies that $\lim_{n\to\infty} \int f_n = \int f$

Proof: Similar to the proof given last time.

Let $\epsilon > 0$ be given, then since $m(E) < \infty$, by Egorov, there is $A_\epsilon \subseteq E$ with $f_n \to f$ uniformly on A_ϵ and $m(E - A_\epsilon) < \epsilon$.

By uniform conv, there is N such that if $n > N$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in A_\epsilon$, but then

$$\begin{aligned} \int_E |f - f_n| \, dx &= \left(\int_{A_\epsilon} + \int_{E - A_\epsilon} \right) |f_n - f| \\ &= \int_{A_\epsilon} \underbrace{|f_n(x) - f(x)|}_{< \epsilon} dx + \int_{E - A_\epsilon} \underbrace{|f_n(x) - f(x)|}_{\leq 2M} dx \\ &\leq m(A_\epsilon)\, \epsilon + 2M\, m(E - A_\epsilon) \end{aligned}$$

$$\leq m(E)\epsilon + 2M\epsilon = \epsilon\left(m(E) + 2M\right) \quad \square$$

This is the first of 3 convergence theorems that we'll see in this course.

3. Riemann vs. Lebesgue Integral

As an interlude, let's compare the Riemann and Lebesgue integrals and show that the Lebesgue one is strictly better:

Definition ϕ is a **step function** if

$$\phi(x) = \sum_{k=1}^{N} a_k \, 1_{R_k} \qquad R_k \text{ rectangle}$$

(It's simple function but the base is a rectangle)

Theorem: If f is Riemann integrable on $[a,b]$, then f is measurable

$$\text{and } \int_{[a,b]}^{\mathcal{R}} f(x)dx = \int_{[a,b]}^{\mathcal{L}} f(x)dx$$

Proof:

STEP 1: By definition, a Riemann integrable function is bounded, say $|f(x)| \leq M$

Riemann integration says there is a sequence $\{\phi_k\}$ (lower function) and $\{\psi_k\}$ (upper function) of step functions bounded by M such that

$$\phi_1(x) \leq \phi_2(x) \leq \cdots \leq f \leq \cdots \leq \psi_2(x) \leq \psi_1(x)$$

$$\lim_{k\to\infty} \int_{[a,b]}^{\mathcal{R}} \phi_k(x)dx = \lim_{k\to\infty} \int_{[a,b]}^{\mathcal{R}} \psi_k(x)dx =: \int_{[a,b]}^{\mathcal{R}} f(x)dx$$

Upshot: Since step functions *are* simple functions, by **LEVEL 1** of the Lebesgue integral, we have

$$\int_{[a,b]}^{\mathcal{R}} \phi_k(x)dx = \int_{[a,b]}^{\mathcal{L}} \phi_k(x)dx \text{ and same for } \psi_k$$

STEP 2: Notice that $\phi_k \to \widetilde{\phi}$ for some $\widetilde{\phi}$ (since ϕ_k increases) and $\psi_k \to \widetilde{\psi}$ for some $\widetilde{\psi}$ (since ψ_k decreases). Hence, since ϕ_k and ψ_k are bounded, by the BCT, we get

$$\lim_{k\to\infty} \int_{[a,b]}^{\mathcal{L}} \phi_k(x)dx = \int_{[a,b]}^{\mathcal{L}} \widetilde{\phi}(x)dx \text{ and same for } \psi_k$$

$$\text{Hence } \int_{[a,b]}^{\mathcal{L}} \widetilde{\phi}(x)dx = \lim_{k\to\infty} \int_{[a,b]}^{\mathcal{L}} \phi_k(x)dx = \lim_{k\to\infty} \int_{[a,b]}^{\mathcal{R}} \phi_k(x)dx =: \int_{[a,b]}^{\mathcal{R}} f(x)dx$$

$$\text{Therefore } \int_{[a,b]}^{\mathcal{L}} \widetilde{\psi} - \widetilde{\phi} = \left(\int_{[a,b]}^{\mathcal{R}} f\right) - \left(\int_{[a,b]}^{\mathcal{R}} f\right) = 0$$

But since $\psi_k - \phi_k \geq 0$, we get $\widetilde{\psi} - \widetilde{\phi} \geq 0$ and so the integral being zero implies that $\widetilde{\psi} = \widetilde{\phi}$ a.e. (see homework), but since $\phi_k \leq f \leq \psi_k$ we get $\widetilde{\phi} \leq f \leq \widetilde{\psi}$ and so $\widetilde{\psi} = \widetilde{\phi} = f$ a.e.

Since $\widetilde{\phi}$ is measurable, being a limit of (measurable) step functions, it follows that f is measurable ✓

STEP 3: Finally, since $\phi_k \to f$, once again by the BCT we have

$$\int_{[a,b]}^{\mathcal{L}} f(x)dx \overset{\text{BCT}}{=} \lim_{k\to\infty} \int_{[a,b]}^{\mathcal{L}} \phi_k(x)dx = \int_{[a,b]}^{\mathcal{R}} f(x)dx \quad \square$$

Non-Example: Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ if rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Then f is not Riemann integrable on $[0,1]$ since the upper-sums are always 1 and the lower sums are always 0.

However, notice $f(x) = 1_E$ where $E = [0,1] \backslash \mathbb{Q}$ and so

$$\int f = m(E) = m([0,1]) - m(\mathbb{Q} \cap [0,1]) = 1 - 0 = 1$$

So the Lebesgue integral is strictly better than the Riemann integral.

4. Level 3: Non-Negative Functions

In general, if $f \geq 0$ is measurable (could be $\pm\infty$), then we define the Lebesgue integral simply as:

Definition:

$$\int f(x)dx = \sup_g \int g(x)dx$$

Where the sup is taken over all g from **LEVEL 2** with $0 \leq g \leq f$, that is g is bounded and supported on a set of finite measure.

Definition: f is **integrable** if $\int |f(x)| \, dx < \infty$

(We can remove the absolute value here since f is non-negative)

Example:

$$f_a(x) = \begin{cases} \frac{1}{|x|^a} & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases} \quad \text{and} \quad g_a(x) = \begin{cases} 0 & \text{if } |x| \leq 1 \\ \frac{1}{|x|^a} & \text{if } |x| > 1 \end{cases}$$

f_a is integrable when $a < d$ and g_a is integrable when $a > d$ (see HW)

Once again, the same properties (linearity, additivity, monotonicity) hold. The only non-obvious part is the following part of linearity:

$$\textbf{Fact: } \int f + g = \int f + \int g$$

Proof: $\boxed{\geq}$ Suppose $\phi \leq f$ and $\psi \leq g$ where ϕ and ψ are bounded and supported on a set of finite measure, then so is $\phi + \psi$ and $\phi + \psi \leq f + g$ and therefore by definition of $\int f + g$ we have

$$\int f + g \geq \int \phi + \psi \overset{\textbf{LEVEL 2}}{=} \int \phi + \int \psi$$

Taking the sup over $\phi \leq f$, we get

$$\int f + g \geq \left(\int f \right) + \int \psi$$

And taking the sup over $\psi \leq g$ we get

$$\int f + g \geq \int f + \int g \checkmark$$

$\boxed{\leq}$ Suppose η is bounded and supported on a set of finite measure with $\eta \leq f + g$. Let $\eta_1(x) =: \min(f(x), \eta(x))$ and $\eta_2 =: \eta - \eta_1$ (bounded and supported on sets of finite measure), and note that $\eta_1 \leq f$ and $\eta_2 \leq g$

$$\int \eta \overset{\textbf{LEVEL 2}}{=} \int \eta_1 + \eta_2 = \int \eta_1 + \int \eta_2 \overset{\text{DEF}}{\leq} \int f + \int g$$

Taking the sup over $\eta \leq f + g$ gives the desired inequality.

Fact: If f is integrable and $0 \leq g \leq f$ then g is integrable

Follows because $\int |g| \leq \int |f| < \infty$

Fact: If f is integrable then $f(x) < \infty$ for almost every x

Proof: Let $E_k = \{x \mid f(x) \geq k\}$ and $E_\infty = \{x \mid f(x) = \infty\}$ then

$$\int_{\mathbb{R}^d} f \geq \int_{E_k} f \geq \int_{E_k} k = km(E_k)$$

Hence $m(E_k) \leq \frac{1}{k}\left(\int f\right) \to 0$, hence $\lim_{k\to\infty} m(E_k) = 0$ but since $E_k \searrow E_\infty$ and $m(E_k) < \infty$, we get $m(E_\infty) = 0$ (from homework)

5. Fatou's Lemma

Recall: If $f_n \to f$ then $\int f_n \not\to \int f$. But what *is* true is that $\int f$ is always smaller:

Fatou's Lemma If $f_n \geq 0$ and $f_n \to f$ pointwise a.e. then

$$\int f \leq \liminf_{n\to\infty} \int f_n$$

So $\int f$ is always smaller than the smallest possible limit of $\int f_n$

Application: This is **INCREDIBLY** useful in the calculus of variations and PDE, which deals with minimizing integrals. Usually, the best you can do is to find sequence f_n of minimizers that converges to some f. Fatou says that $\int f$ is **even smaller** than all the $\int f_n$ (in the liminf sense), and so f is usually the minimizer you're looking for!

Proof: We want to use the **LEVEL 3** definition of the integral: Suppose $0 \leq g \leq f$ where g is bounded and supported on some E with

$m(E) < \infty$.

Let $g_n(x) =: \min(g(x), f_n(x))$ then $g_n \to g$ a.e. so by the BCT we have

$$\int g_n \to \int g$$

By construction $g_n \leq f_n$ and so $\int g_n \leq \int f_n$ and so taking $\liminf$ on both sides we get $\liminf_{n\to\infty} \int g_n \leq \liminf_{n\to\infty} \int f_n$ and so

$$\int g = \lim_{n\to\infty} \int g_n = \liminf_{n\to\infty} \int g_n \leq \liminf_{n\to\infty} \int f_n$$

$$\int g \leq \liminf_{n\to\infty} \int f_n$$

Finally, taking the sup over g yields the result □

6. The Monotone Convergence Theorem

We are now ready to prove our second convergence theorem:

Definition: $f_n \nearrow f$ if $f_n(x) \leq f_{n+1}(x)$ for all n and $f_n \to f$ a.e.

Monotone Convergence Theorem: If $f_n \geq 0$ with $f_n \nearrow f$ then

$$\lim_{n\to\infty} \int f_n = \int f$$

Notice how few assumptions there are here!!

Proof: Since $f_n(x) \leq f(x)$ a.e. we necessarily have $\int f_n \leq \int f$ and taking $\limsup$ we get

$$\limsup_{n\to\infty} \int f_n \leq \int f$$

But then by Fatou we get

$$\int f \leq \liminf_{n \to \infty} \int f_n \leq \limsup_{n \to \infty} \int f_n \leq \int f$$

Which shows that

$$\lim_{n \to \infty} \int f_n = \int f \quad \square$$