LECTURE 22: DOUBLE INTEGRALS (II)

1. Double Integrals over general regions

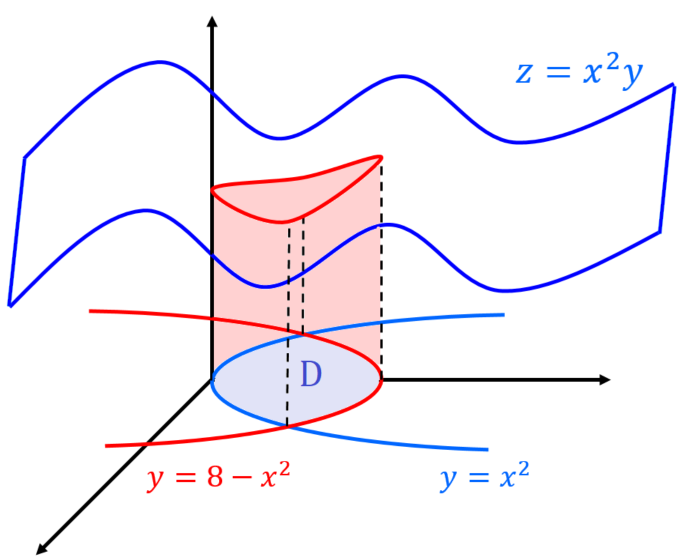
Video: Double Integrals over General Regions

So far we have done the case where the base of our cake is a rectangle, but now we can also do the one where the base is any general region.

Example 1:

Find $\int\int_D x^2y\,dxdy$ where D is the region between the curves $y = x^2$ and $y = 8 - x^2$

STEP 1: Picture (important)

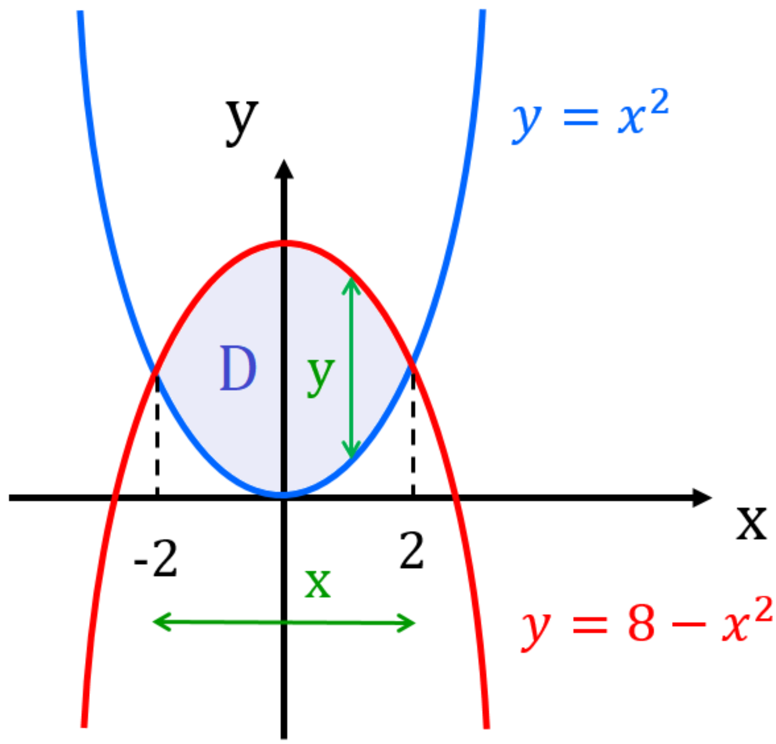


Date: Monday, October 18, 2021.

Here D stands for domain of integration; it's the base of our cake and the space where x and y lie in.

Analogy: Here we want to find the volume of the cake whose base is D and whose height is $z = x^2y$

STEP 2: Draw D



STEP 3: Find inequalities for x and y

This is the trickiest part, here is where our picture helps

Useful Trick: For double integration problems, it's useful to use:

$$\text{Smaller } \leq y \leq \text{ Bigger}$$
$$x^2 \leq y \leq 8 - x^2$$

For x, notice that the bounds are given precisely by the points of intersection of the parabolas:

Points of Intersection:

$$\begin{aligned} x^2 &= 8 - x^2 \\ 2x^2 &= 8 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Therefore we get:

$$\begin{cases} x^2 \leq y \leq 8 - x^2 \\ -2 \leq x \leq 2 \end{cases}$$

STEP 4: Integrate

$$\begin{aligned} &\int\int_D x^2 y \, dx dy \\ =&\int_{-2}^{2} \left(\int_{x^2}^{8-x^2} x^2 y \, dy \right) dx \\ =&\int_{-2}^{2} \left[x^2 \left(\frac{1}{2} y^2 \right) \right]_{y=x^2}^{y=8-x^2} dx \\ =&\int_{-2}^{2} \frac{1}{2} x^2 \left(8 - x^2 \right)^2 - \frac{1}{2} x^2 \left(x^2 \right)^2 dx \\ =&\int_{-2}^{2} \frac{1}{2} x^2 \left(8 - x^2 \right)^2 - \frac{1}{2} x^6 dx \\ =&2 \int_{0}^{2} \frac{1}{2} x^2 \left(8 - x^2 \right)^2 - \frac{1}{2} x^6 dx \qquad \text{(The integrand is even)} \\ =&\int_{0}^{2} x^2 \left(8 - x^2 \right)^2 - x^6 dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^2 x^2\left(64 - 16x^2 + x^4\right) - x^6 dx \\
&= \int_0^2 64x^2 - 16x^4 + \cancel{x^6} - \cancel{x^6} dx \\
&= \left[\frac{64}{3}x^3 - \frac{16}{5}x^5\right]_0^2 \\
&= \frac{64}{3} \times 8 - \frac{16}{5} \times 32 \\
&= \frac{512}{3} - \frac{512}{5} \\
&= 512\left(\frac{1}{3} - \frac{1}{5}\right) \\
&= 512\left(\frac{5-3}{15}\right) \\
&= \frac{1024}{15}
\end{aligned}
$$

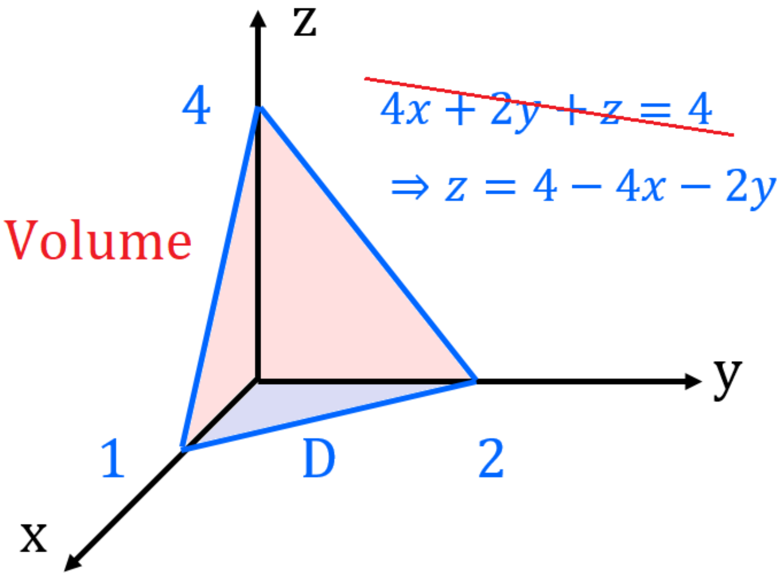
Note: This seems very daunting at first, but it's always the same technique: Draw a picture, find inequalities, and integrate.

⚠ Make sure the outer integral only has constants! For instance, your integral shouldn't be $\int_{x^2}^{8-x^2}\int_{-2}^{2} x^2 y$

Example 2:

Find the volume of the tetrahedron under $4x + 2y + z = 4$ in the first octant.

STEP 1: Picture:



Note: *Could* use the normal vector to draw the tetrahedron, but here is is much easier to do with intercepts:

Intercepts:

$z-$Intercept $(x = 0, y = 0)$: $4(0) + 2(0) + z = 4 \Rightarrow z = 4$

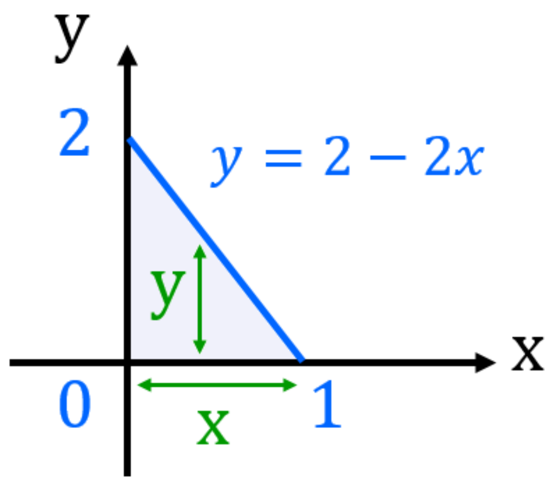
$y-$Intercept $(x = 0, z = 0)$: $4(0) + 2y + 0 = 4 \Rightarrow 2y = 4 \Rightarrow y = 2$

$x-$Intercept $(y = 0, z = 0)$: $4x + 2(0) + 0 = 4 \Rightarrow 4x = 4 \Rightarrow x = 1$

STEP 2:

$$\text{Volume } = \int\int_D 4 - 4x - 2y \, dx dy$$

STEP 3: Inequalities



To find the equation of the line, either find the equation of the line going through $(1, 0)$ and $(0, 2)$, or (easier) notice $z = 0$ in D, and hence:

$$\begin{aligned} &4x + 2y + 0 = 4 \\ \Rightarrow &2y = 4 - 4x \\ \Rightarrow &y = 2 - 2x \end{aligned}$$

For the inequalities, as before, use

$$\begin{aligned} \text{Smaller } \leq y \leq & \text{ Bigger} \\ 0 \leq y \leq & \, 2 - 2x \\ 0 \leq x \leq & \, 1 \end{aligned}$$

STEP 4: Integrate

$$\begin{aligned}
&\text{Volume} \\
=&\int_0^1 \int_0^{2-2x} 4 - 4x - 2y \, dy dx \\
=&\int_0^1 \left[(4-4x)y - y^2\right]_{y=0}^{y=2-2x} dx \\
=&\int_0^1 (4-4x)(2-2x) - (2-2x)^2 - 0 + 0 dx \\
=&\int_0^1 (4)(1-x)(2)(1-x) - 2^2(1-x)^2 dx \\
=&\int_0^1 8(1-x)^2 - 4(1-x)^2 dx \\
=&\int_0^1 4(x-1)^2 dx \\
=&\left[\frac{4}{3}(x-1)^3\right]_0^1 \\
=&\frac{4}{3}
\end{aligned}$$

2. Horizontal Regions

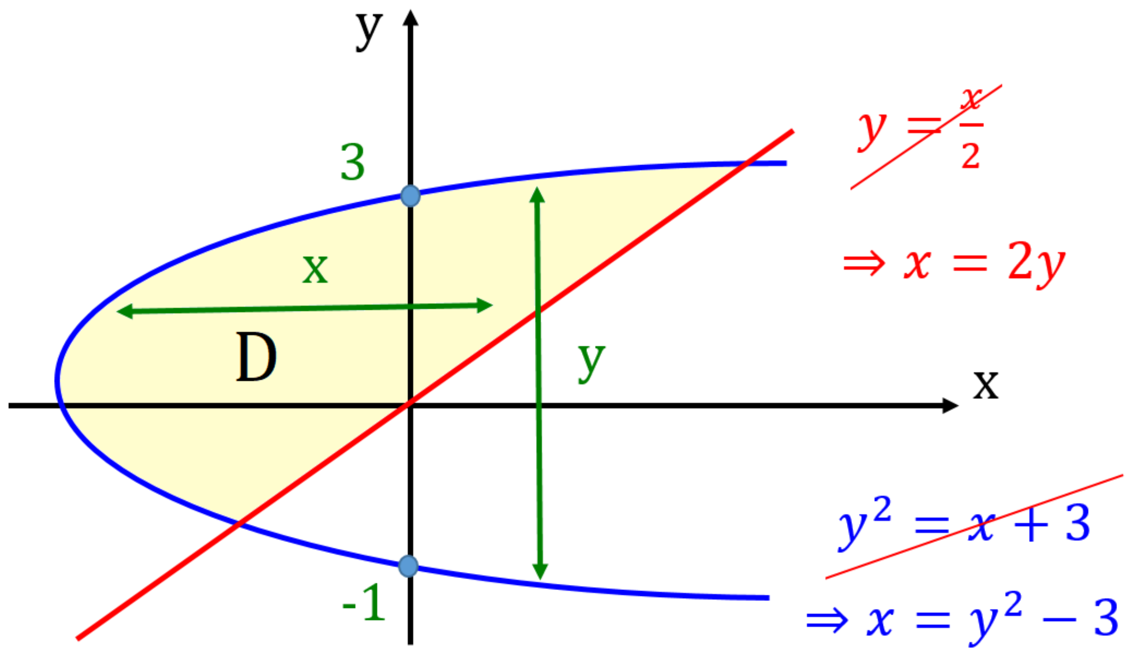
Example 3:

Calculate the following integral, where D is the region between $y = \frac{x}{2}$ and $y^2 = x + 3$

$$\int \int_D xy dx dy$$

STEP 1: Picture

(To draw this, notice $y^2 = x + 3 \Rightarrow x = y^2 - 3$ and tilt your head)



STEP 2: Inequalities

Here D is a "horizontal" region, so have to do x first:

Note: For horizontal regions, get

$$\text{Left} \leq x \leq \text{Right}$$
$$y^2 - 3 \leq x \leq 2y$$

$$? \leq y \leq ?$$

Points of intersection

$$\begin{aligned} 2y &= y^2 - 3 \\ y^2 - 2y - 3 &= 0 \\ (y+1)(y-3) &= 0 \\ y &= -1, 3 \end{aligned}$$

Inequalities

$$\begin{aligned} y^2 - 3 \leq x \leq 2y \\ -1 \leq y \leq 3 \end{aligned}$$

STEP 3: Integrate

$$\begin{aligned} \int\int_D xy dx dy &= \int_{-1}^{3} \int_{y^2-3}^{2y} xy \, dx dy \\ &= \int_{-1}^{3} \left[\left(\frac{x^2}{2} \right) y \right]_{x=y^2-3}^{x=2y} dy \\ &= \int_{-1}^{3} \frac{(2y)^2 y}{2} - \frac{\left(y^2-3\right)^2 y}{2} dy \\ &= \int_{-1}^{3} \frac{\left(4y^2\right) y}{2} - \frac{1}{2}\left(y^4 - 6y^2 + 9\right) y \, dy \\ &= \int_{-1}^{3} 2y^3 - \frac{y^5}{2} + 3y^3 - \frac{9}{2} y \, dy \\ &= \int_{-1}^{3} 5y^3 - \frac{y^5}{2} - \frac{9}{2} y \, dy \end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{5}{4}y^4 - \frac{y^6}{12} - \frac{9y^2}{4}\right]_{-1}^{3} \\
&= \frac{5}{4}\left(3^4 - (-1)^4\right) - \frac{1}{12}\left(3^6 - (-1)^6\right) - \frac{9}{4}\left(3^2 - (-1)^2\right) \\
&= \frac{5}{4} \times 80 - \frac{1}{12} \times 728 - \frac{9}{4} \times 8 \\
&= 100 - \frac{182}{3} - 18 \\
&= 82 - \frac{182}{3} \\
&= \frac{246 - 182}{3} \\
&= \frac{64}{3}
\end{aligned}
$$

3. An Impossible Integral?

Video: Changing Order of Integration

Let's use the idea above to calculate an impossible integral!

Example 4:

Calculate the following integral:

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$$

Note: This integral is impossible to evaluate directly, since $\sin\left(y^2\right)$ doesn't have an anti-derivative! Instead, use the following:

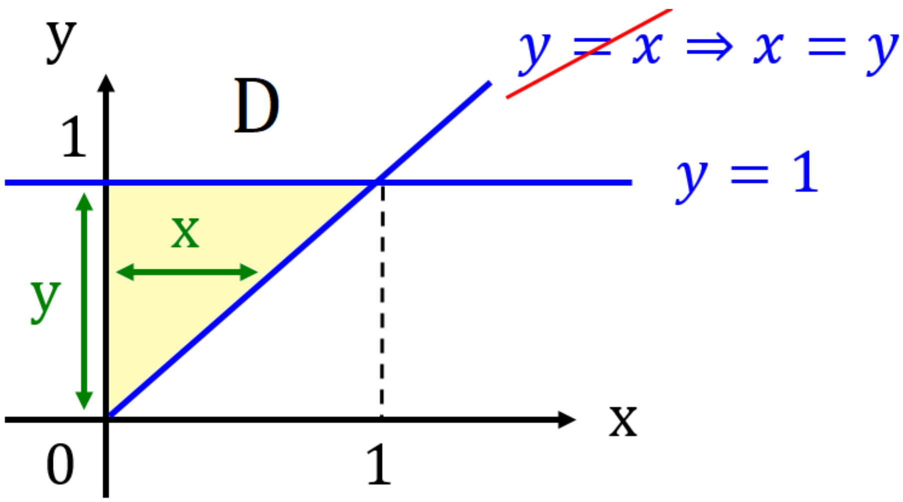
Trick: Change the order of x and y

STEP 1: Find D

Here the region is

$$\begin{aligned} x \leq y \leq 1 \\ 0 \leq x \leq 1 \end{aligned}$$

Picture:



STEP 2: Write D as a horizontal region

$$\begin{aligned} \text{Left } \leq x \leq \text{ Right} \\ 0 \leq x \leq y \\ 0 \leq y \leq 1 \end{aligned}$$

STEP 3: Integrate

$$\begin{aligned}
&\int_0^1 \int_x^1 \sin(y^2) dy dx \\
=&\int_0^1 \int_0^y \sin(y^2) dx dy \\
=&\int_0^1 \left[\sin\left(y^2\right) x\right]_{x=0}^{x=y} dy \\
=&\int_0^1 \sin\left(y^2\right) y dy \\
&\left(u = y^2, du = 2y dy \Rightarrow y = \frac{1}{2} du, u(0) = 0, u(1) = 1\right) \\
=&\int_0^1 \sin(u) \left(\frac{1}{2} du\right) \\
=&\left[-\frac{1}{2}\cos(u)\right]_0^1 \\
=&\frac{1}{2} - \frac{1}{2}\cos(1)
\end{aligned}$$

Note: For another fun application of this Fubini trick, check out the following video, sometimes called a Frullani Integral:

Video: Multivariable Integral