LECTURE 22: DOMINATED CONVERGENCE

1. Convergence of Series

Recall: Monotone Convergence Theorem: If $f_n \geq 0$ and $f_n \nearrow f$

$$\text{Then } \lim_{n\to\infty} \int f_n = \int f$$

As a corollary, let's prove some useful facts about convergence of series of functions:

Corollary: Consider a series $\sum_{k=1}^{\infty} a_k(x)$ where $a_k \geq 0$ is measurable

$$\text{Then } \int \sum_{k=1}^{\infty} a_k(x)dx = \sum_{k=1}^{\infty} \int a_k(x)dx$$

Proof: Let $f_n(x) = \sum_{k=1}^{n} a_k(x)$ then $f_n \nearrow f$ where $f(x) =: \sum_{k=1}^{\infty} a_k(x)$ and so by MCT we get

$$\lim_{n\to\infty} \int f_n = \int f$$

$$\text{Hence } \sum_{k=1}^{\infty} \int a_k(x)dx \overset{\text{DEF}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} \int a_k(x)dx \overset{\text{FINITE}}{=} \lim_{n\to\infty} \int \sum_{k=1}^{n} a_k(x)dx$$

$$\overset{\text{DEF}}{=} \lim_{n\to\infty} \int f_n = \int f \overset{\text{DEF}}{=} \int \sum_{k=1}^{\infty} a_k(x)dx \quad \square$$

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Important Corollary: If $\sum_{k=1}^{\infty} \int a_k(x)dx$ is finite, then $\sum_{k=1}^{\infty} a_k(x)$ converges for a.e. x

Why? In that case, $f(x) = \sum_{k=1}^{\infty} a_k(x)$ is integrable and hence $f(x)$ is finite a.e. that is, $\sum_{k=1}^{\infty} a_k(x)$ is finite a.e.

In other words, if you can integrate a series and the result is converges, then the original series converges for every x. This is a useful way (especially in physics) to show that a series of functions converges a.e.

2. Level 4: General Case

LEVEL 4: For general f, just write $f = f^+ - f^-$ where

$$f^+ = \max(f(x), 0) \text{ and } f^- = \max(-f(x), 0)$$

Here $f^\pm$ are non-negative functions and so

Definition:

$$\int f =: \int f^+ - \int f^-$$

Definition: f is **integrable** if $\int |f(x)|\, dx < \infty$.

This is equivalent to requiring that $f^\pm$ is integrable.

Note: $\int f$ is independent of the decomposition used:

If $f = f_1 - f_2 = g_1 - g_2$ where the f_i and g_i are non-negative integrable functions, then $f_1 + g_2 = g_1 + f_2$ and so from **LEVEL 3**

$$\int f_1 + \int g_2 = \int g_1 + \int f_2 \Rightarrow \int f_1 - \int f_2 = \int g_1 - \int g_2$$

Fact: All the facts discussed before (linearity, additivity, monotonicity, triangle intequality) are true here as well.

Note: You can also define the Lebesgue integral for complex-valued functions: If $f(x) = u(x) + i\,v(x)$ is complex valued, then

$$\int f =: \int u + i \int v$$

Moreover, f is integrable if $\int |f| < \infty$, where $|f| = \sqrt{u^2 + v^2}$ is the modulus of f

3. Regularity

In order to prepare for the celebrated Dominated Convergence Theorem, let's prove some regularity properties of integrable functions.

WARNING: If f is integrable, we do **NOT** have $\lim_{|x|\to\infty} f(x) = 0$ (see homework). That said, we do have the following fact:

Fact 1: If f is integrable, then for all $\epsilon > 0$ there is a ball B such that

$$\int_{B^c} |f(x)|\, dx < \epsilon$$

So intuitively, integrable functions are small at ∞

Proof: WLOG, assume $f \geq 0$, because otherwise replace f with $|f|$.

$$\text{Let } f_n = f \;\, 1_{B(0,n)}$$

Then $f_n \geq 0$ measurable and $f_n \nearrow f$, so by the Monotone Converge Theorem,

$$\lim_{n\to\infty} \int f_n = \int f \Rightarrow \lim_{n\to\infty} \int f - f_n = 0$$

Let $\epsilon > 0$ be given, then by the above there is N large enough so that

$$\int f - f_N < \epsilon$$

But $f - f_N = f - f\,1_{B(0,N)} = f\left(1 - 1_{B(0,N)}\right) = f\,1_{B(0,N)^c}$

Let $B = B(0,N)$ then the above implies that

$$\int_{B^c} f < \epsilon \quad \square$$

Fact 2: If f is integrable, then for all $\epsilon > 0$ there is $\delta > 0$ such that

$$\text{If } m(E) < \delta \text{ then } \int_E |f(x)|\,dx < \epsilon$$

This is like the $\epsilon - \delta$ definition of continuity, but for measures

Proof: Again, assume $f \geq 0$ and let $\epsilon > 0$ be given.

Let $f_n =: f\,1_{E_n}$ where $E_n = \{x \mid f(x) \leq n\}$ and notice $f_n \leq n$

Then $f_n \geq 0$ measurable and $f_n \nearrow f$, so by the Monotone Convergence Theorem we have $\lim_{n\to\infty} \int f - f_n = 0$, so there is $N > 0$ such that

$$\int f - f_N < \frac{\epsilon}{2}$$

Let δ TBA, then if $m(E) < \delta$, then

$$\begin{aligned}\int_E f &= \int_E \underbrace{f - f_N}_{\geq 0} + \int_E f_N \\ &\leq \int_{\mathbb{R}^d} f - f_N + \int_E \underbrace{f_N}_{\leq N} \\ &\leq \frac{\epsilon}{2} + N m(E) \\ &< \frac{\epsilon}{2} + N\delta\end{aligned}$$

If you choose δ such that $N\delta < \frac{\epsilon}{2}$ then you get $\int_E f < \epsilon$ □

4. The Dominated Convergence Theorem

Video: Dominated Convergence Theorem

We are now ready to prove the cornerstone theorem of Lebesgue integration: the Dominated Convergence Theorem. It can be viewed as a culmination of our efforts, and is a general statement about the interchange of limits and integrals.

Dominated Convergence Theorem: Suppose $\{f_n\}$ is a sequence such that $f_n \to f$ a.e. and $|f_n| \leq g$ where g is an integrable function

$$\text{Then } \lim_{n\to\infty} \int |f_n - f| = 0$$

Note: It's like the bounded convergence theorem, except we replace M with any integrable function g. So if f_n is dominated by g where g (independent of n) is integrable, then we can interchange limits and integrals.

Proof: Consider $E_n =: \{|x| \leq n \text{ and } g(x) \leq n\}$.

Given $\epsilon > 0$ then repeating the proof of Fact 1, there is M such that

$$\int_{E_M^c} g < \epsilon$$

The functions $f_n \, 1_{E_M}$ converge to f on E_M, are bounded by M (because $|f_n| \leq g \leq M$ on E_M), and supported on a set of finite measure, so by the Bounded Convergence Theorem, there is N such that if $n > N$

$$\int_{E_M} |f_n - f| < \epsilon$$

With the same N, if $n > N$ we get

$$\begin{aligned}\int |f_n - f| = & \int_{E_M} |f_n - f| + \int_{E_M^c} \underbrace{|f_n - f|}_{\leq 2g} \\ \leq & \epsilon + 2\int_{E_M^c} g \\ \leq & \epsilon + 2\epsilon = 3\epsilon \quad \square\end{aligned}$$

5. The space L^1 of integrable functions

The space of integrable functions has a particularly nice structure.

Definition: $L^1(\mathbb{R}^d)$ = space of integrable functions

Definition: If f is integrable, then the L^1 **norm** of f is

$$\|f\| = \|f\|_{L_1} =: \int |f(x)| \, dx$$

You can check that this is a norm in the usual sense. For example we have $\|f+g\| \leq \|f\| + \|g\|$

Norms allow us to define the distance between two integrable functions f and g as

$$d(f,g) = \|g - f\|$$

This then defines a metric on L^1, and in fact:

Theorem: [Riesz-Fischer] (L^1, d) is complete

This makes Lebesgue integrals drastically different from Riemann integrals. The space of Riemann integrable functions is incomplete. For example $f_n = \min(n, -\ln(x))$ is Cauchy but doesn't converge.

Proof:

STEP 1: Suppose $\{f_n\}$ is Cauchy in L^1, that is $\|f_n - f_m\|$ goes to 0 as $m, n \to \infty$. The plan is to extract a subsequence of $\{f_n\}$ that converges to some f pointwise *and* in the norm. This can be achieved if the convergence is fast enough.

STEP 2: **Claim # 1:** There is subsequence $\{f_{n_k}\}$ such that

$$\left\| f_{n_{k+1}} - f_{n_k} \right\| \leq 2^{-k}$$

Proof of Claim: You do this inductively. Suppose you found f_{n_k}, then by Cauchiness with $\epsilon = 2^{-k}$ there is $N = N(2^{-k})$ such that if $n \geq N$ then

$$\|f_n - f_{n_k}\| \leq 2^{-k}$$

Then just let $n_{k+1} = N$ ✓

STEP 3: Our function f

$$\text{Define: } f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x)$$

$$\text{And } g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right|$$

$$\begin{aligned} \int g\,dx &= \int |f_{n_1}| + \int \sum_{k=1}^{\infty} \left| f_{n_{k+1}} - f_{n_k} \right| dx \\ &= \int |f_{n_1}| + \sum_{k=1}^{\infty} \int \left| f_{n_{k+1}} - f_{n_k} \right| dx \\ &\leq \int |f_{n_1}| + \sum_{k=1}^{\infty} 2^{-k} \\ &= \int |f_{n_1}| + 1 < \infty \end{aligned}$$

(The interchange of series and integrals is justified by the Series Fact from today)

Hence g is integrable, and since $|f| \leq g$, this implies f is integrable.

In particular, the series defining f converges almost everywhere, and since the partial sums of that series are precisely f_{n_k} (telescoping series), we find that $f_{n_k} \to f$ a.e. x

STEP 4: Claim # 2: $f_{n_k} \to f$ in L^1

This follows because each partial sum is dominated by g. Therefore, by the Dominated Convergence Theorem, we get $\|f_{n_k} - f\| \to 0$

STEP 5: Claim # 3: $f_n \to f$ in L^1

Just need to use Cauchiness: Given $\epsilon > 0$ there is N such that for all $m, n > N$ then $\|f_n - f_m\| < \frac{\epsilon}{2}$. If n_k (for k large enough) is chosen such that $n_k > N$ and $\|f_{n_k} - f\| < \frac{\epsilon}{2}$ (from **STEP 4**) then if $n > N$ we have

$$\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence f_n converges to f in L^1 □

In the proof, we have shown the following fact:

Corollary: If $f_n \to f$ in L^1 then there is a subsequence f_{n_k} such that $f_{n_k} \to f$ a.e.

6. L^p SPACES

Similarly you can define L^p with $1 \leq p < \infty$ as

Definition: $f \in L^p$ if $\int |f(x)|^p < \infty$ and

$$\|f\|_{L_p} = \left(\int |f(x)|^p \right)^{\frac{1}{p}}$$

With a similar proof, you can show that L^p is complete.

The space L^2 is particularly noteworthy because it is a Hilbert space, that is there is an inner product

$$(f, g) = \int f(x)\overline{g(x)}dx$$

Whose norm $\|f\|_{L_2} = \sqrt{(f,f)}$ makes L^2 complete

The case $p = \infty$ is defined a bit differently:

Definition: $f \in L^\infty$ if there is a C such that $|f(x)| \leq C$ for a.e. x

$$\|f\|_{L^\infty} = \inf \{C \text{ such that } |f(x)| \leq C \text{ for a.e. } x\}$$

Those are called the essentially bounded functions.

Here L^∞ is complete as well, but with a different proof.