LECTURE 23: L'HÔPITAL'S RULE

Today: We'll prove *the* most addictive theorem: L'Hôpital's Rule.

1. L'Hôpital's Rule

Video: Proof of L'Hôpital's Rule

Intuitively, it just says that:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Here we will prove the $\frac{0}{0}$ case

L'Hôpital's Rule

Suppose f and g are differentiable with

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

Moreover, suppose

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$$

$$\text{Then: } \lim_{x \to a} \frac{f(x)}{g(x)} = L$$

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Note: All those assumptions are important, and in fact there are counterexamples if some of them don't hold (see HW)

To simplify the proof, assume $g(x) \neq 0$ and $g'(x) \neq 0$

Proof:

STEP 1: GOAL

Let $\epsilon > 0$ be given.

We want to find $\delta > 0$ such that if $0 < |x - a| < \delta$, then $\left|\frac{f(x)}{g(x)} - L\right| < \epsilon$

STEP 2:

Fix x

Let t be TBA (More variables = More freedom in our proof)

Intuitively, if $f(t)$ and $g(t)$ are small, then:

$$\frac{f(x)}{g(x)} \approx \frac{\frac{f(x)-f(t)}{x-t}}{\frac{g(x)-g(t)}{x-t}} = \frac{f(x)-f(t)}{g(x)-g(t)}$$

Trick:

$$\begin{aligned}\left|\frac{f(x)}{g(x)} - L\right| &= \left|\frac{f(x)}{g(x)} - \left(\frac{f(x)-f(t)}{g(x)-g(t)}\right) + \left(\frac{f(x)-f(t)}{g(x)-g(t)}\right) - L\right| \\ &\leq \underbrace{\left|\frac{f(x)}{g(x)} - \left(\frac{f(x)-f(t)}{g(x)-g(t)}\right)\right|}_{A} + \underbrace{\left|\left(\frac{f(x)-f(t)}{g(x)-g(t)}\right) - L\right|}_{B}\end{aligned}$$

We want to show this is $< \epsilon$, so let's study each piece separately.

STEP 3: Study of A

$$
\begin{aligned}
A &= \left| \frac{f(x)}{g(x)} - \left(\frac{f(x) - f(t)}{g(x) - g(t)} \right) \right| \\
&= \frac{|f(x)(g(x) - g(t)) - (f(x) - f(t))g(x)|}{|g(x)(g(x) - g(t))|} \\
&= \frac{|\cancel{f(x)g(x)} - f(x)g(t) - \cancel{g(x)f(x)} + g(x)f(t)|}{|g(x)|\,|g(x) - g(t)|} \\
&= \frac{|-f(x)g(t) + g(x)f(t)|}{|g(x)|\,|g(x) - g(t)|} \qquad \text{(Triangle Inequality)} \\
&\leq \frac{|f(x)|\,|g(t)| + |g(x)|\,|f(t)|}{|g(x)|\,|g(x) - g(t)|} \qquad \text{(Looks like FOIL)} \\
&\leq \frac{|f(x)|\,|f(t)| + |f(x)|\,|g(t)| + |g(x)|\,|f(t)| + |g(x)|\,|g(t)|}{|g(x)|\,|g(x) - g(t)|} \\
&= \frac{(|f(x)| + |g(x)|)\,(|f(t)| + |g(t)|)}{|g(x)|\,|g(x) - g(t)|} \\
&= \left(\frac{|f(x)| + |g(x)|}{|g(x)|\,|g(x) - g(t)|} \right) \underbrace{(|f(t)| + |g(t)|)}_{\text{Good}}
\end{aligned}
$$

Notice that the $|f(t)| + |g(t)|$ term is good/small, since $f(t), g(t) \to 0$

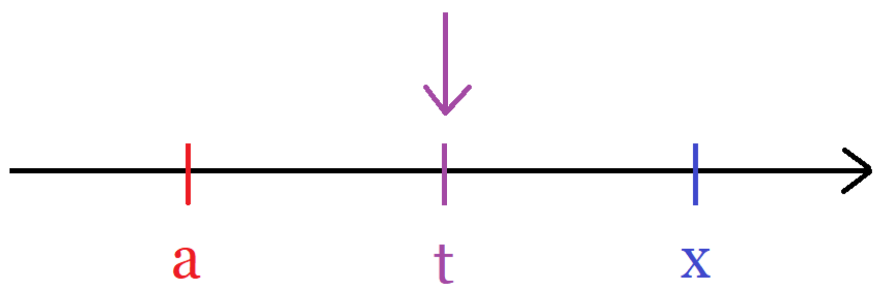
The idea now is to choose t that cancels the numerator $|f(x)| + |g(x)|$ and the denominator $|g(x)|$:

What is t? Given x, let t be such that $a < t < x$ and

$$(1)\ |f(t)| + |g(t)| < \left(\frac{|g(x)|^2}{4\left(|f(x)| + |g(x)|\right)} \right) \epsilon$$

(2) TBA

We can do that since x is fixed and $|f(t)| + |g(t)| \to 0$



(t is like an advance guard for x, protects x from a)

So by (1) and our estimate of A, we get:

$$A \leq \left(\frac{\cancel{|f(x)| + |g(x)|}}{\cancel{|g(x)|}\, |g(x) - g(t)|} \right) \left(\frac{|g(x)|^{\cancel{2}}}{4\left(\cancel{|f(x)| + |g(x)|}\right)} \right) \epsilon = \frac{\epsilon}{4} \left(\frac{|g(x)|}{|g(x) - g(t)|} \right)$$

To estimate $|g(x) - g(t)|$ we need to use the reverse triangle inequality, so assume that

$$(2)\ |g(t)| < \frac{|g(x)|}{2}$$

Then: $|g(x) - g(t)| \geq \big||g(x)| - |g(t)|\big| \geq |g(x)| - |g(t)| \overset{(2)}{>} |g(x)| - \dfrac{|g(x)|}{2} = \dfrac{|g(x)|}{2}$

$$\text{Therefore: } A \leq \frac{\epsilon}{4}\left(\frac{|g(x)|}{|g(x)-g(t)|}\right) < \frac{\epsilon}{4}\cancel{|g(x)|}\frac{2}{\cancel{|g(x)|}} = \frac{\epsilon}{2}$$

STEP 4: Study of B

$$\text{Recall: } B = \left|\frac{f(x)-f(t)}{g(x)-g(t)} - L\right|$$

Idea: By the Mean-Value Theorem, for some c and d

$$\frac{f(x)-f(t)}{g(x)-g(t)} = \frac{(f(x)-f(t))/(x-t)}{(g(x)-g(t))/(x-t)} = \frac{f'(c)}{g'(d)}$$

Here c and d are **NOT** necessarily equal. That said, **IF** $c = d$, then:

$$\frac{f(x)-f(t)}{g(x)-g(t)} = \frac{f'(c)}{g'(c)} \to L \qquad \text{as } c \to a$$

Which would imply that $B = \left|\frac{f(x)-f(t)}{g(x)-g(t)} - L\right|$ is small.

To fix this, we need a new and improved version of the MVT called:

Ratio MVT

There exists c in (x, t) such that:

$$\frac{f(x)-f(t)}{g(x)-g(t)} = \frac{f'(c)}{g'(c)}$$

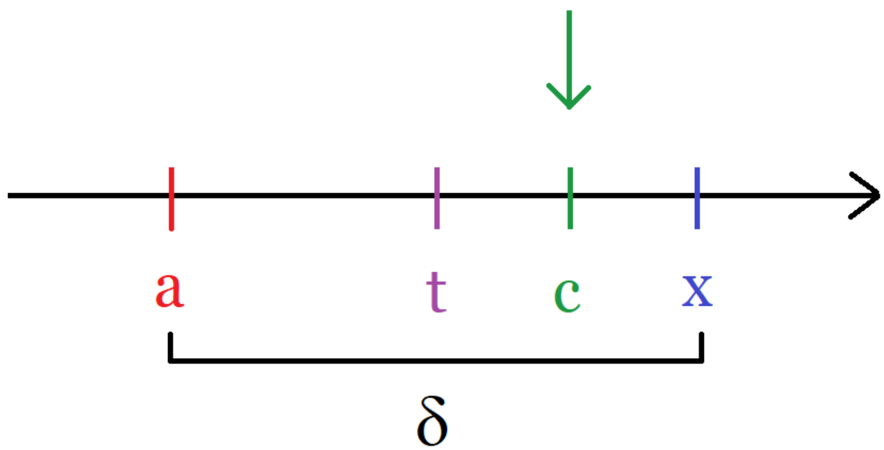
(The proof is in the appendix)

Note: If $g(x) = x$, then we get:

$$\frac{f(x)-f(t)}{x-t} = f'(c) \text{ which is the MVT}$$

In this case, using the ratio MVT, we get:

$$B = \left| \frac{f(x) - f(t)}{g(x) - g(t)} - L \right| = \left| \frac{f'(c)}{g'(c)} - L \right|$$



Since $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ (we never used this), there is $\delta > 0$ such that for all x, if $0 < |x - a| < \delta$, then $\left| \frac{f'(x)}{g'(x)} - L \right| < \frac{\epsilon}{2}$

But since $|c - a| < |x - a| < \delta$, we can use the above with c instead of x to conclude that:

$$\Rightarrow B = \left| \frac{f'(c)}{g'(c)} - L \right| < \frac{\epsilon}{2}$$

STEP 5: Grand Finale!

With that $\delta > 0$, if $|x - a| < \delta$, we then get

$$\left| \frac{f(x)}{g(x)} - L \right| \leq A + B < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \square$$

Remarks:

(1) The proof of the $\frac{\infty}{\infty}$ case is similar to this one, except that we have an advance guard $a < x < t$ instead of a rear guard.

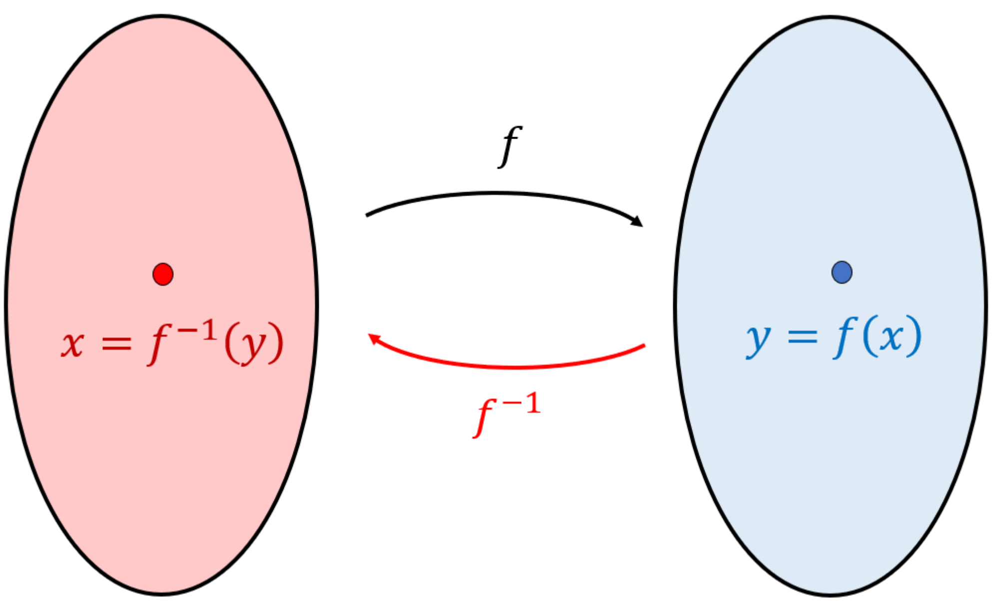
(2) The proof of the case $x \to \infty$ follows from this one simply by letting $h = \frac{1}{x} \to 0^+$ and therefore:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{h\to 0^+} \frac{f\left(\frac{1}{h}\right)}{g\left(\frac{1}{h}\right)} \stackrel{\hat{H}}{=} \lim_{h\to 0^+} \frac{f'\left(\frac{1}{h}\right)\left(-\frac{1}{h^2}\right)}{g\left(\frac{1}{h}\right)\left(-\frac{1}{h^2}\right)} = \lim_{h\to 0^+} \frac{f'\left(\frac{1}{h}\right)}{g'\left(\frac{1}{h}\right)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

2. The Inverse Function Theorem

Finally, let's find the derivative of f^{-1}

This allows us to find the derivatives of $\ln(x), \sqrt{x}, \tan^{-1}(x)$, etc.



Motivation: Suppose $f(x) = y$, so $x = f^{-1}(y)$. Start with $f\left(f^{-1}(y)\right) = y$ and differentiate both sides:

$$\begin{aligned}\left[f\left(f^{-1}(y)\right)\right]' &= (y)' \\ f'\left(f^{-1}(y)\right)\left(f^{-1}(y)\right)' &= 1 \\ \left(f^{-1}(y)\right)' &= \frac{1}{f'\left(f^{-1}(y)\right)} \\ \left(f^{-1}(y)\right)' &= \frac{1}{f'(x)}\end{aligned}$$

This motivates the following theorem:

Inverse Function Theorem

Suppose f is one-to-one and differentiable at x_0. If $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}(y_0)\right)' = \frac{1}{f'(x_0)}$$

Example 1:

Let $f(x) = \tan(x)$, then $f^{-1}(y) = \tan^{-1}(y)$, and the above theorem says

$$\left(\tan^{-1}(y)\right)' = \frac{1}{(\tan)'(x)} = \frac{1}{\sec^2(x)} = \frac{1}{1+\tan^2(x)} = \frac{1}{1+\tan^2(\tan^{-1}(y))} = \frac{1}{1+y^2}$$

Hence $\left(\tan^{-1}(x)\right)' = \frac{1}{1+x^2}$

Example 2:

Let $f(x) = x^2$, then $f^{-1}(y) = \sqrt{y}$, and the above theorem says

$$\left(\sqrt{y}\right)' = \frac{1}{(x^2)'} = \frac{1}{2x} = \frac{1}{2\sqrt{y}} \Rightarrow \left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$$

Inverse Function Theorem Proof:

STEP 1: By definition we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

By taking reciprocals, which is valid since $f'(x_0) \neq 0$ and $f(x) \neq f(x_0)$ (f is one-to-one), we get:

$$\lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$$

STEP 2: Let $\epsilon > 0$ be given, then the above limit says: there is $\delta' > 0$ such that if $0 < |x - x_0| < \delta'$, then

$$\left| \frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \epsilon$$

But f^{-1} is continuous at $y_0 = f(x_0)$, so by definition of continuity (with δ' instead of ϵ), there is $\delta > 0$ such that if $0 < |y - y_0| < \delta$, then

$$\left| \underbrace{f^{-1}(y)}_{x} - \underbrace{f^{-1}(y_0)}_{x_0} \right| < \delta' \Rightarrow |x - x_0| < \delta'$$

STEP 3: But then, if $0 < |y - y_0| < \delta$, we get $|x - x_0| < \delta'$ and so

$$\left|\frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)}\right| < \epsilon$$
$$\left|\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} - \frac{1}{f'(x_0)}\right| < \epsilon$$

This shows that

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}$$

$$\text{That is: } \left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} \quad \square$$

Cultural Note: There is also something called the Implicit Function Theorem, which allows you to differentiate functions that are defined implicitly, like $x^3 y + y^2 x = 2$

3. Appendix: Proof of Ratio MVT

Ratio MVT

There exists c in (x, t) such that:

$$\frac{f(x) - f(t)}{g(x) - g(t)} = \frac{f'(c)}{g'(c)}$$

Proof: Define

$$h(s) = (f(x) - f(t))(g(s) - g(x)) - (g(x) - g(t))(f(s) - f(x))$$

Then:

$$h(x) = (f(x) - f(t))(g(x) - g(x)) - (g(x) - g(t))(f(x) - f(x)) = 0$$
$$h(t) = (f(x) - f(t))(g(t) - g(x)) - (g(x) - g(t))(f(t) - f(x)) = 0$$

So by Rolle's there is c in (x, t) such that $h'(c) = 0$. However

$$\begin{aligned} h'(s) &= (f(x) - f(t))\, g'(s) - (g(x) - g(t))\, f'(s) \\ \Rightarrow h'(c) &= (f(x) - f(t))\, g'(c) - (g(x) - g(t))\, f'(c) = 0 \\ &\Rightarrow (f(x) - f(t))\, g'(c) = (g(x) - g(t))\, f'(c) \\ &\Rightarrow \frac{f(x) - f(t)}{g(x) - g(t)} = \frac{f'(c)}{g'(c)} \quad \square \end{aligned}$$