LECTURE 23: DOUBLE INTEGRALS (III)

1. Volumes

Finally, let's discuss some useful properties of double integrals. This first one should be familiar:

Interpretation:

$\int\int_D f(x,y)dxdy$ is the volume under the graph of f and over D

Use this to your advantage when calculating hard integrals:

Example 1:

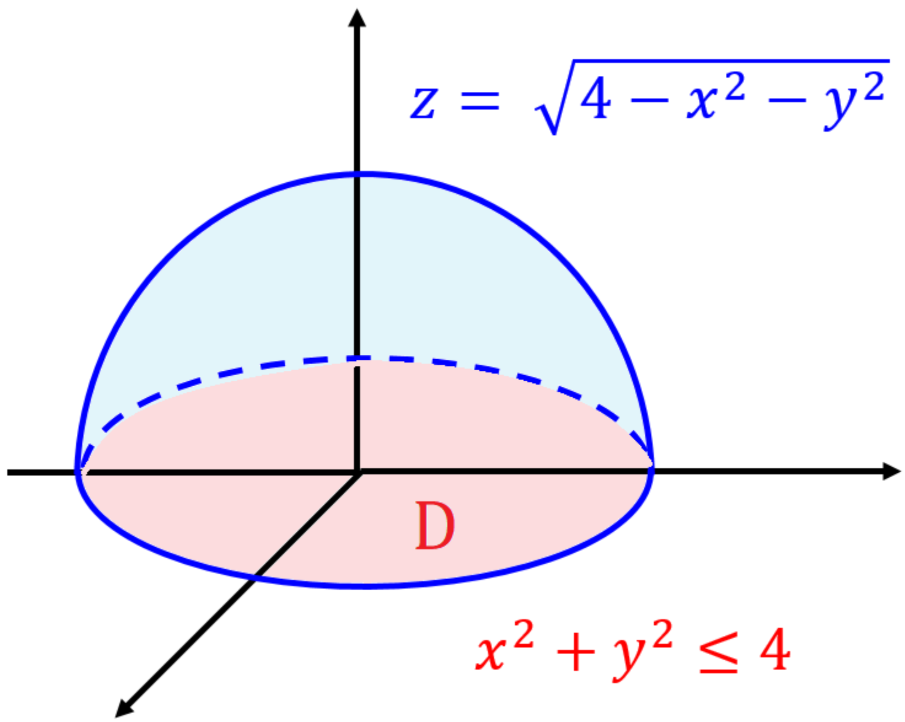
Calculate the following integral, where D is the disk $x^2 + y^2 \leq 4$

$$\int\int_D \sqrt{4 - x^2 - y^2}dxdy$$

$$z = \sqrt{4 - x^2 - y^2} \Rightarrow z^2 = 4 - x^2 - y^2 \Rightarrow x^2 + y^2 + z^2 = 4 \text{ and } z \geq 0$$

Hence $\sqrt{4 - x^2 - y^2}$ represents the upper hemisphere

Date: Wednesday, October 20, 2021.



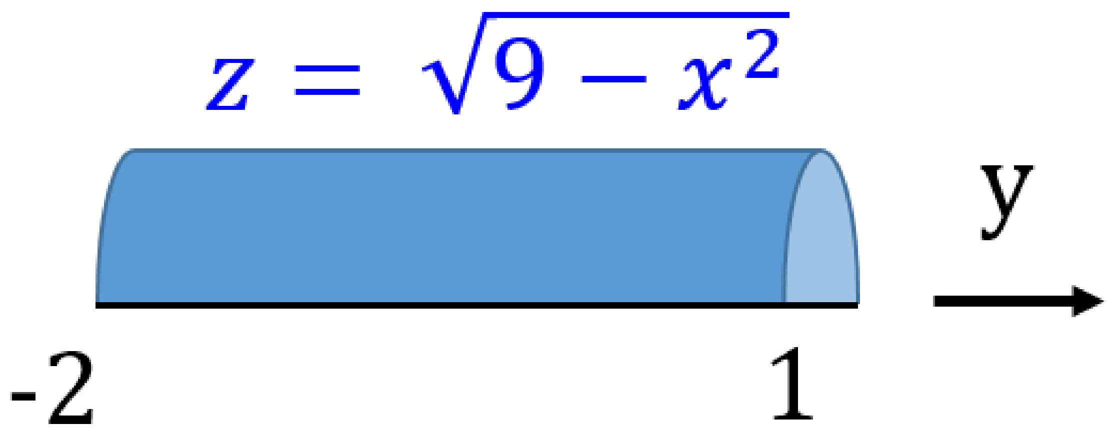
$$\int\int_D \sqrt{4 - x^2 - y^2}dxdy = \text{ Vol of upper-hemisphere } = \frac{1}{2}\left(\frac{4}{3}\pi\,(2)^3\right) = \frac{16}{3}\pi$$

Example 2:

Evaluate the following integral:

$$\int_{-2}^{1}\int_{-3}^{3} \sqrt{9 - x^2}dxdy$$

$z = \sqrt{9 - x^2} \Rightarrow x^2 + z^2 = 9$ and $z \geq 0 \Rightarrow$ Upper cylinder in y direction



$$\int_{-2}^{1}\int_{-3}^{3}\sqrt{9-x^2}\,dxdy = \frac{1}{2}\pi(3)^2(1-(-2)) = \frac{27}{2}\pi$$

Physical Application:

If $f(x,y)$ is the density of a metal plate D, then $\int\int_D f(x,y)dxdy$ gives you the mass of D

2. Average Value

Recall:

The average value of $f(x)$ over $[a,b]$ is

$$\frac{\int_a^b f(x)dx}{b-a} = \frac{\int_a^b f(x)dx}{\text{Length of } [a,b]}$$

Definition:

The average value of $f(x,y)$ over D is

$$\frac{\int\int_D f(x,y)dxdy}{\text{Area of D}}$$

Example 3:

Find the average value of $f(x,y) = \sqrt{4-x^2-y^2}$ over the disk $D: x^2+y^2 \leq 4$

$$\int\int_D f(x,y)dxdy = \int\int_D \sqrt{4-x^2-y^2}dxdy = \frac{16}{3}\pi \text{ (see Example above)}$$

$$\text{Area of } D = \pi(2)^2 = 4\pi$$

Therefore the average value is

$$\frac{\frac{16}{3}\pi}{4\pi} = \frac{16}{4(3)} = \frac{4}{3}$$

3. Review: Riemann Sums

Now that we know how to evaluate double integrals, let's go back and actually *define* what a double integral is.

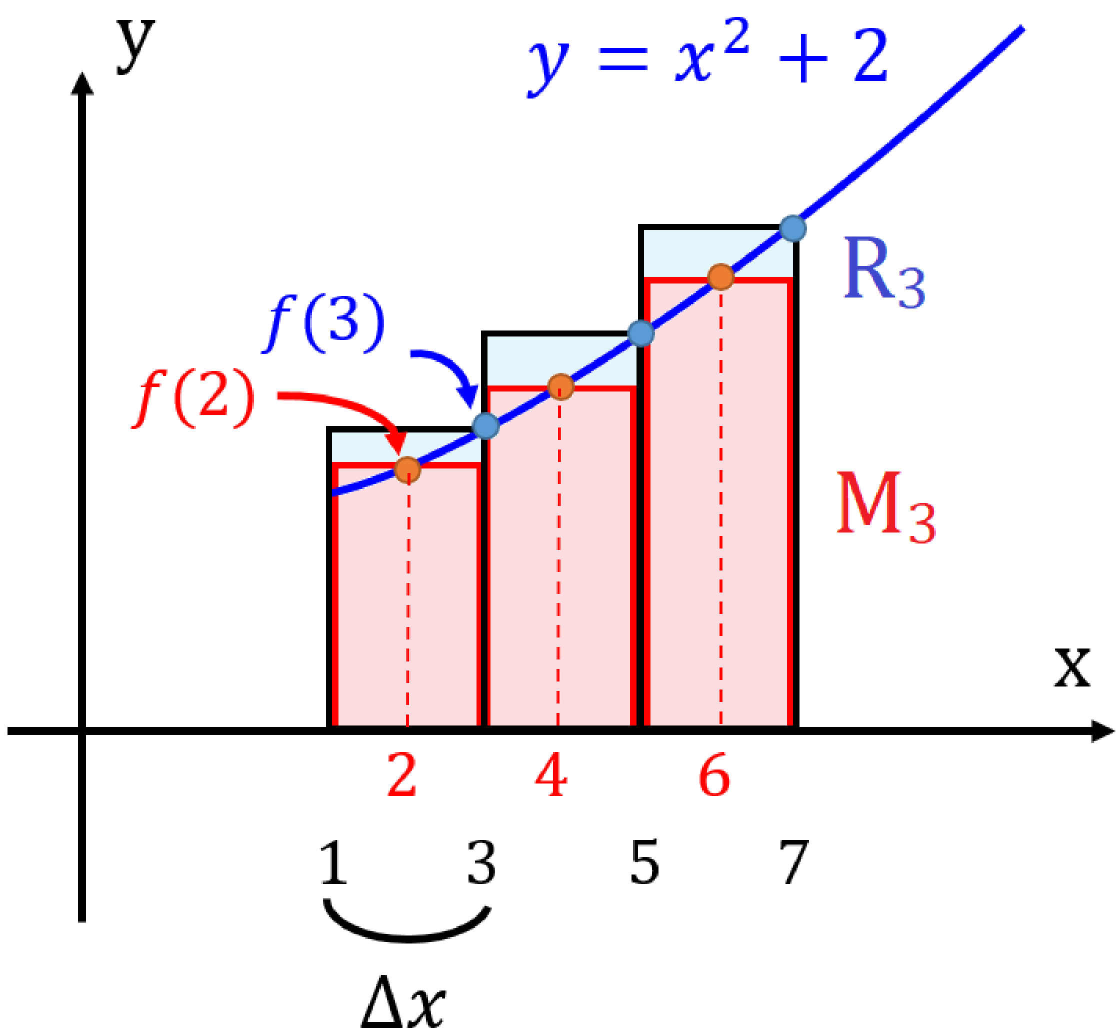
Let's first review Riemann sums from single-variable calculus:

Example 4:

Estimate the area under $f(x) = x^2 + 2$ from 1 to 7

(a) Using $n = 3$ rectangles and right endpoints

(b) Using $n = 3$ and midpoints



(a) First find the width:

$$\Delta x = \frac{7-1}{n} = \frac{6}{3} = 2$$

Consider the rectangle with height $f(\text{Right})$. The sum of the areas becomes:

$$R_3 = (\Delta x)\left(f(3) + f(5) + f(7)\right) = 2\left(11 + 27 + 51\right) = 178$$

(b) Now we take the rectangles with height $f(\text{Midpoint})$

$$M_3 = 2(f(2) + f(4) + f(6)) = 124$$

4. Riemann Sums in 2 Dimensions

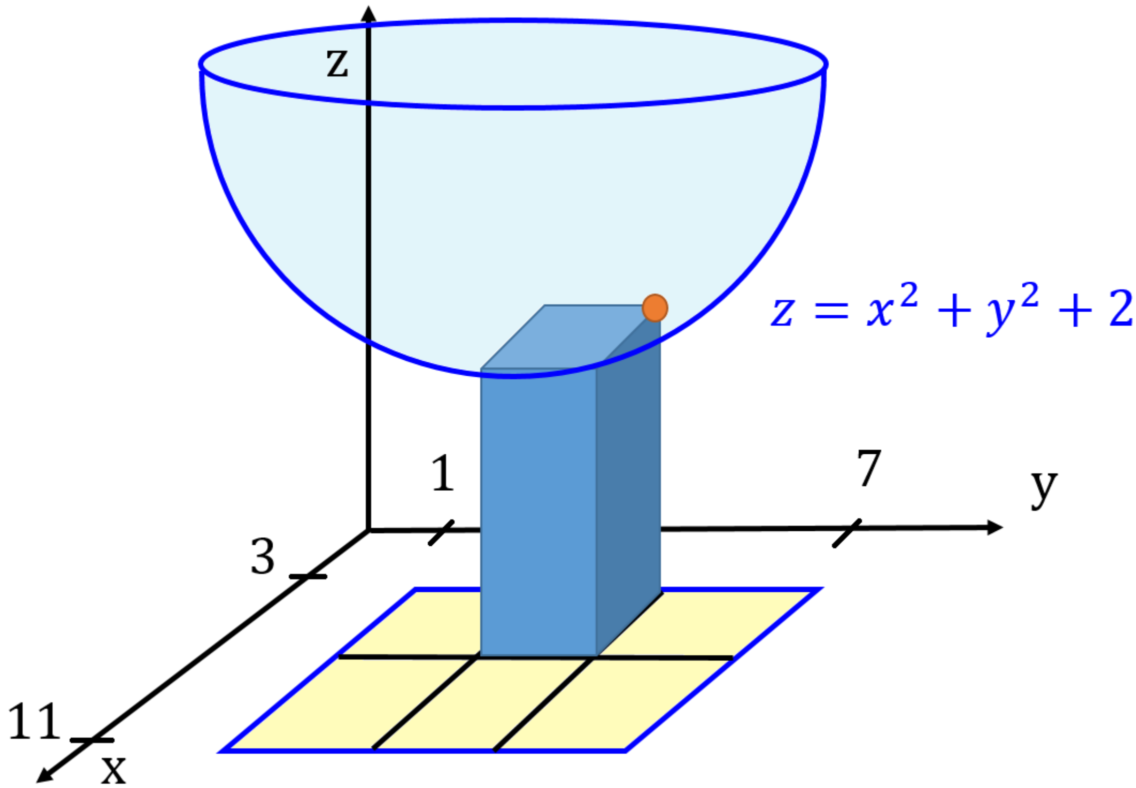
In two dimensions, it's the same thing, except rectangles become boxes

Example 5:

Estimate the volume under $f(x,y) = x^2 + y^2 + 2$ on $[3,11] \times [1,7]$

(a) Using $m = 2$ and $n = 3$ and upper right points

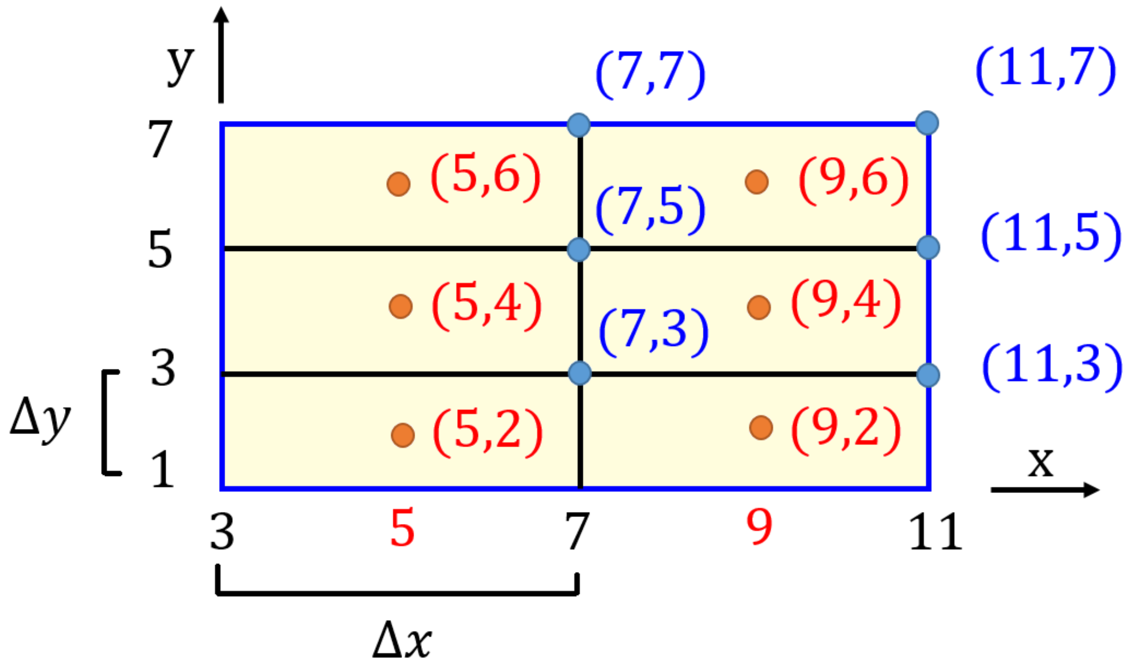
(b) Using $m = 2$ and $n = 3$ and midpoints



(a) This time we would like to approximate the volume with boxes, so first sub-divide the rectangle $[3, 11] \times [1, 7]$ into sub-rectangles:

Width:

$$\Delta x = \frac{11 - 3}{m} = \frac{8}{2} = 4$$
$$\Delta y = \frac{7 - 1}{n} = \frac{6}{3} = 2$$



There are 6 rectangles in total, and on each rectangle, consider the box with height $f(\text{Upper Right Point})$, and we get

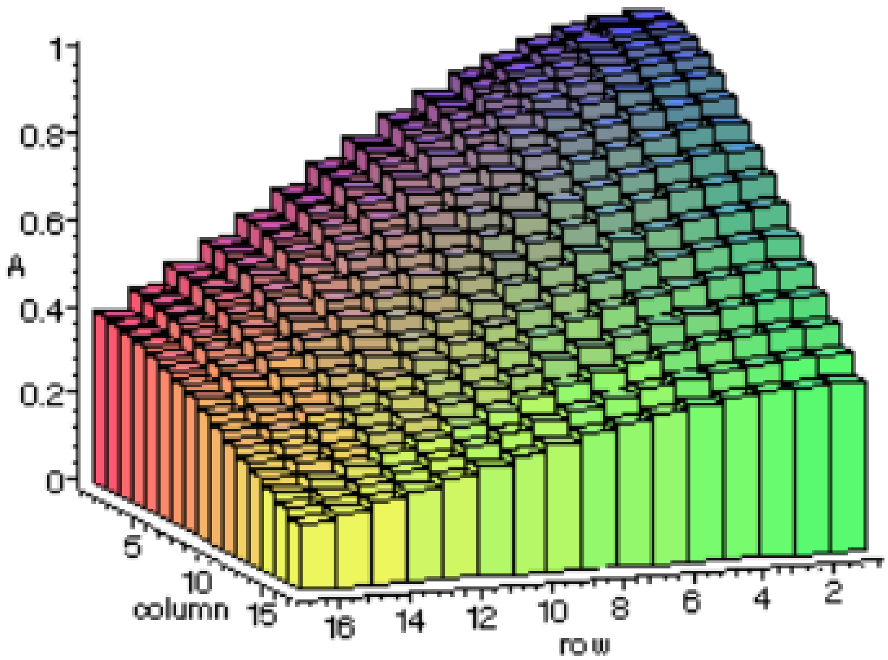
$$\begin{aligned} R_{2,3} =& \underbrace{(\Delta x)}_{4}\underbrace{(\Delta y)}_{2}\left(f(7,3)+f(7,5)+f(7,7)+f(11,3)+f(11,5)+f(11,7)\right) \\ =& 8 \times 688 \\ =& 5504 \end{aligned}$$

(Notice this is a *double sum*, sometimes abbreviated as $\sum\sum$)

(b) Same but this time you choose $f(\text{Midpoint})$ and get

$$M_{2,3} = 8\left(f(5,2)+f(5,4)+f(5,6)+f(9,2)+f(9,4)+f(9,6)\right) = 3536$$

Note: A double integral is just that, but you let m and n go to ∞.



Note: The picture above is taken from this website.

5. Other Properties

Finally, here are some other miscellaneous properties that double integrals enjoy:

(1) Areas:

Recall:

$$\int_a^b 1\,dx = b - a = \text{ Length of } [a,b]$$

Areas:

$$\int\int_D 1dxdy = \text{ Area of } D$$

(2) Linearity

Linearity:

$$\int\int f+g = \int\int f + \int\int g$$
$$\int\int cf = c\int\int f \text{ } (c \text{ is a constant})$$

In particular, we get

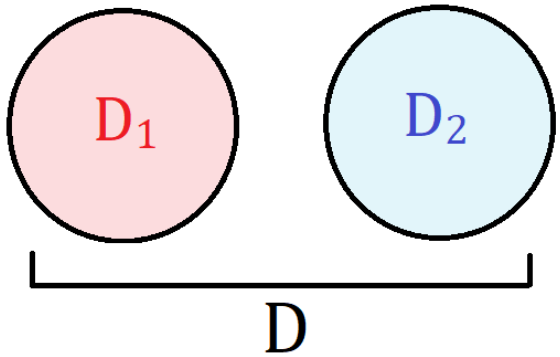
$$\int\int_D c = c\int\int_D 1 = c \text{ Area of } (D)$$

(3) Splitting Regions:

Splitting:

If D is made out of 2 pieces D_1 and D_2 as in the picture below, then

$$\int\int_D f = \int\int_{D_1} f + \int\int_{D_2} f$$



(4) Comparison Property

Comparison:

$$f \leq g \Rightarrow \int\int_D f \leq \int\int_D g$$

(This is basically saying that integrating doesn't change the order)

In particular, we get the following fact:

$$f \geq 0 \Rightarrow \int\int_D f \geq 0$$

So for example, if on the homework or exams you find that

$$\int\int_D \sqrt{4 - x^2 - y^2}\, dxdy = -1$$

You definitely made a mistake since that integral should be positive!

Moreover, we obtain that:

$$f \leq g \leq h \Rightarrow \int\int_D f \leq \int\int_D g \leq \int\int_D h$$

So integration preserves the order. The following problem is a nice illustration of this:

Example 6: (extra practice)

Estimate the following integral, where D is the disk $x^2 + y^2 \leq 9$:

$$\int\int_D e^{-(x^2+y^2)} dxdy$$

Start with:

$$0 \leq x^2 + y^2 \leq 9$$
$$-9 \leq -(x^2 + y^2) \leq 0$$
$$e^{-9} \leq e^{-(x^2+y^2)} \leq e^0 = 1$$
$$\int\int_D e^{-9} dxdy \leq \int\int_D e^{-(x^2+y^2)} dxdy \leq \int\int_D 1 dxdy$$

However:

$$\int\int_D 1\, dxdy = \text{Area of } D = \pi(3^2) = 9\pi$$
$$\int\int_D e^{-9}\, dxdy = e^{-9} \text{ (Area of } D) = e^{-9}(9\pi) = \frac{9\pi}{e^9}$$

Therefore we get:

$$\frac{9\pi}{e^9} \leq \int\int_D e^{-(x^2+y^2)} dxdy \leq 9\pi$$

In other words, the integral is roughly in the interval $[0.0034, 28.27]$