LECTURE 23: FINAL EXAM REVIEW

1. Dominated Convergence

Problem 1: Suppose $\int |g(x)|\, dx < \infty$ and $|f'|$ is bounded, show that

$$\lim_{h \to 0} \int g(x) \left(\frac{f(x+h) - f(x)}{h} \right) dx = \int g(x) f'(x) dx$$

Solution: To use the Dominated Convergence Theorem, we just need to show that the left function is dominated by an integrable function.

$$\left| g(x) \left(\frac{f(x+h) - f(x)}{h} \right) \right| = |g(x)| \left| \frac{f(x+h) - f(x)}{h} \right| = |g(x)| \, |f'(c)| \leq C \, |g(x)|$$

Here we used the Mean-Value Theorem

Therefore, since $C\,|g|$ is integrable, by the Dominated Convergence Theorem, we have

$$\lim_{h \to 0} \int g(x) \left(\frac{f(x+h) - f(x)}{h} \right) dx = \int \lim_{h \to 0} g(x) \left(\frac{f(x+h) - f(x)}{h} \right) dx = \int g f' dx$$

2. Chebyshev's Inequality

Problem 2: Suppose $f \geq 0$ and f is integrable. Show that if $t > 0$ then

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$$m\{x \mid f(x) > t\} \leq \frac{1}{t}\left(\int f\right)$$

Solution:

$$\begin{aligned}
\int f &= \int_{\{x \mid f(x) > t\}} f + \int_{\{x \mid f(x) \leq t\}} f \\
&\geq \int_{\{x \mid f(x) > t\}} f \\
&\geq \int_{\{x \mid f(x) > t\}} t \\
&= tm\{x \mid f(x) > t\}
\end{aligned}$$

$$\text{Hence } tm\{x \mid f(x) > t\} \leq \int f$$

And dividing by t gives the result

3. L^p SPACES

Problem 3: Show that if $m(E) < \infty$ and $f \in L^2(E)$ then $f \in L^1(E)$ and give a counterexample for $E = \mathbb{R}$

Solution: The idea is to use that if $f \geq 1$ then $f \leq f^2$ and hence

$$\begin{aligned}
\int_E |f|\, dx &= \int_{\{|f| \geq 1\}} |f|\, dx + \int_{\{|f| < 1\}} |f|\, dx \\
&\leq \int_{\{|f| \geq 1\}} |f|^2\, dx + \int_{\{|f| < 1\}} 1 dx \\
&\leq \int_E |f|^2\, dx + \int_E 1 dx \\
&= \int_E |f|^2\, dx + m(E) \\
&< \infty
\end{aligned}$$

For the counterexample, let

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Then } \int f^2 = \int_1^\infty \frac{1}{x^2} dx = 1 < \infty$$

$$\text{But } \int |f| = \int_1^\infty \frac{1}{x} dx = \infty$$

Problem 4: Show that there is a sequence $\{f_n\}$ with $f_n \in L^1$ and a function f such that $f_n \to f$ in L^1 but $f_n(x) \to f(x)$ for no x

Solution: Define $f = 0$ and

$$f_1 = 1_{[0,1]}$$
$$f_2 = 1_{\left[0,\frac{1}{2}\right]}\ f_3 = 1_{\left[\frac{1}{2},1\right]}$$
$$f_4 = 1_{\left[0,\frac{1}{4}\right]}, f_5 = 1_{\left[\frac{1}{4},\frac{1}{2}\right]}, f_6 = 1_{\left[\frac{1}{2},\frac{3}{4}\right]}, f_7 = 1_{\left[\frac{3}{4},1\right]}$$

Then $f_n(x) \nrightarrow 0$ for no x (Given x we have $f_n(x) = 1$ for infinitely many n), but

$$\|f_n - f\| = \int_0^1 |f_n(x) - f(x)|\, dx = \int_0^1 |f_n| \to 0$$

Since the areas under f_n shrink to 0.

4. Measurability

Problem 5: If $\{f_n\}$ is a sequence of measurable functions, show that the set of points x at which $\{f_n(x)\}$ converges is measurable.

Solutions: Usually in those measurability questions, it's enough to write your set as a union/intersection of measurable sets.

Here $\{f_n(x)\}$ is Cauchy, meaning for every k there is N such that if $m, n \geq N$ then $|f_m(x) - f_n(x)| < \frac{1}{k}$

Hence the set in question can be written as

$$\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{m,n=N}^{\infty} \left\{ x \text{ such that } |f_m(x) - f_n(x)| < \frac{1}{k} \right\}$$

And therefore we're done because the union/intersection of measurable sets is measurable

5. Derivative

Problem 6: Show that if f is differentiable at x, then so is $|f|^2$ and

$$\left(|f|^2\right)'(x) = 2f(x) \cdot f'(x)$$

Hint: $|f|^2 = f \cdot f$

Solution:

$$\begin{aligned}
&|f|^2(x+h) \\
=& f(x+h) \cdot f(x+h) \\
=& (f(x) + f'(x)h + r(h)) \cdot (f(x) + f'(x)h + r(h)) \\
=& |f|^2(x) + 2f(x) \cdot (f'(x)h) + (f'(x)h) \cdot (f'(x)h) + r(h) \cdot (f(x) + f'(x)h + r(h))
\end{aligned}$$

We're done once we show the h^2 term and the $r(h)$ term are sublinear

$$\lim_{h\to 0}\frac{|(f'(x)h)\cdot(f'(x)h)|}{|h|}\leq \lim_{h\to 0}\frac{|f'(x)h|\,|f'(x)h|}{|h|}\leq \lim_{h\to 0}\frac{\|f'(x)\|\,|h|\,\|f'(x)\|\,|h|}{|h|}$$
$$=\lim_{h\to 0}\|f'(x)\|^2\,|h| = 0$$

$$\lim_{h\to 0}\frac{|r(h)\cdot(f(x)+f'(x)h+r(h))|}{|h|}\leq \lim_{h\to 0}\frac{|r(h)|}{|h|}\left(|f(x)|+\|f'(x)\|\,|h|+|r(h)|\right)$$
$$=0\times\left(|f(x)|+\|f'(x)\|\,0+0\right)=0$$

6. Partial Derivatives

Problem 7: Let $f(0,0)=0$ and

$$f(x,y)=\frac{xy\left(x^2-y^2\right)}{x^2+y^2}$$

Calculate $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$. Does $f'(0,0)$ exist? Why or why not?

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

Here we used $f(x,0)=0$ for every x, and similarly $\frac{\partial f}{\partial y}(0,0)=0$ since $f(0,y)=0$ for every y.

For the differentiability issue, consider

$$r(x,y)=f(x,y)-f(0,0)-f_{x_1}(0,0)x-f_{x_2}(0,0)y=f(x,y)$$

It suffices to show that $\lim_{(x,y)\to(0,0)}\frac{|r(x,y)|}{\sqrt{x^2+y^2}}=0$, but

$$\frac{|r(x,y)|}{\sqrt{x^2+y^2}}=\frac{\left|\frac{xy\left(x^2-y^2\right)}{x^2+y^2}\right|}{\sqrt{x^2+y^2}}=\frac{|x|\,|y|\,\left|x^2-y^2\right|}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

To evaluate this limit, it's easiest to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$ then the above limit as $(x, y) \to (0, 0)$ just becomes

$$\begin{aligned}\lim_{r\to 0}\left|\frac{r\cos(\theta)r\sin(\theta)\left(r^2\cos^2(\theta) - r^2\sin^2(\theta)\right)}{\left(r^2\right)^{\frac{3}{2}}}\right| &= \lim_{r\to 0}\left|\frac{r^4\left(\sin(2\theta)\cos(2\theta)\right)}{r^3}\right| \\ &= \lim_{r\to 0} r\left|\sin(2\theta)\cos(2\theta)\right| = 0\end{aligned}$$

7. Implicit Function Theorem

Problem 8: Consider the system

$$\begin{cases} 3x^2z + 6wy^2 - 2z + 1 = 0 \\ xz - \dfrac{4y}{z} - 3w - z = 0 \end{cases}$$

Show that you can solve for x and y in terms of z and w around the point $(1, 2, -1, 0)$ and calculate $G'(-1, 0)$ (where G is the graph of x, y in terms of z, w)

Solution:

$$F(x, y, z, w) = \begin{bmatrix} 3x^2z + 6wy^2 - 2z + 1 \\ xz - \frac{4y}{z} - 3w - z \end{bmatrix}$$

To use the Implicit Function Theorem, check $\det F_{x,y}(1, 2, -1, 0) \neq 0$ (the derivative with respect to what you want to solve for is nonzero)

$$F_{x,y} = \begin{bmatrix} 6xz & 12wy \\ z & -\frac{4}{z} \end{bmatrix}$$

$$F_{x,y}(1, 2, -1, 0) = \begin{bmatrix} 6(1)(-1) & 12(0)(2) \\ -1 & -\frac{4}{-1} \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ -1 & 4 \end{bmatrix}$$

$$\det F_{x,y}(1, 2, -1, 0) = -6(4) - 0 = -24 \neq 0$$

Therefore the Implicit Function Theorem says that there is G such that $(x, y) = G(z, w)$ near $(1, 2, -1, 0)$. Moreover

$$G'(-1,0) = -\left(F_{x,y}(1,2,-1,0)\right)^{-1}\left(F_{z,w}(1,2,-1,0)\right)$$

$$F_{z,w} = \begin{bmatrix} 3x^2 - 2 & 6y^2 \\ x + \frac{4y}{z^2} - 1 & -3 \end{bmatrix}$$

$$F_{z,w}(1,2,-1,0) = \begin{bmatrix} 3(1)^2 - 2 & 6(2)^2 \\ 1 + \frac{4(2)}{(-1)^2} - 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 24 \\ 8 & -3 \end{bmatrix}$$

$$G'(-1,0) = -\begin{bmatrix} -6 & 0 \\ -1 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 24 \\ 8 & -3 \end{bmatrix} = -\left(-\frac{1}{24}\right)\begin{bmatrix} 4 & 0 \\ 1 & -6 \end{bmatrix}\begin{bmatrix} 1 & 24 \\ 8 & -3 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 4 & 96 \\ -47 & 42 \end{bmatrix}$$

8. Fourier Fun

Problem 9: Let f be the 2π periodic function defined on $[-\pi, \pi]$

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq \delta \\ 0 & \text{if } |x| > \delta \end{cases}$$

(a) Find the Fourier coefficients of f

Case 1: If $n \neq 0$ then

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx =& \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-inx}dx \\ =& \frac{1}{2\pi}\left(\frac{1}{-in}\right)\left(e^{-in\delta} - e^{in\delta}\right) \\ =& \frac{1}{\pi n}\left(\frac{e^{in\delta} - e^{-in\delta}}{2i}\right) \\ =& \frac{\sin(\delta n)}{\pi n} \end{aligned}$$

Case 2: If $n = 0$ then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left(\delta - (-\delta)\right) = \frac{\delta}{\pi}$$

(b) Deduce from Parseval that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}$$

From Parseval, we get

$$\begin{aligned}
\left(\frac{\delta}{\pi}\right)^2 + \sum_{n\neq 0}^{\infty}\left(\frac{\sin(\delta n)}{\pi n}\right)^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x))^2\, dx \\
\frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty}\frac{\sin^2(\delta n)}{\pi^2 n^2} &= \frac{1}{2\pi}\left(\delta - (-\delta)\right) \\
2\sum_{n=1}^{\infty}\frac{\sin^2(\delta n)}{\pi^2 n^2} &= \frac{\delta}{\pi} - \frac{\delta^2}{\pi^2} \\
\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2(\delta n)}{n^2} &= \frac{\delta}{\pi^2}\left(\frac{\pi - \delta}{2}\right) \\
\sum_{n=1}^{\infty}\frac{\sin^2(\delta n)}{n^2\delta} &= \frac{\pi - \delta}{2}
\end{aligned}$$

(c) What happens if you let $\delta = \frac{\pi}{2}$ in the above?

In that case $\sin^2(\delta n) = \sin^2\left(\frac{\pi n}{2}\right)$ which is 0 for even n and 1 for odd n and so the above becomes

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\left(\frac{2}{\pi}\right) = \frac{\pi - \frac{\pi}{2}}{2}$$
$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi}{2}\left(\frac{\pi}{4}\right)$$
$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

(d) Finally, let $\delta \to 0$ and show that

$$\int_0^{\infty} \left(\frac{\sin(x)}{x}\right)^2 dx = \frac{\pi}{2}$$

(Assume the integral is well-defined and converges)

$$\int_0^{\infty} \left(\frac{\sin(x)}{x}\right)^2 dx = \lim_{A\to\infty} \int_0^{A} \left(\frac{\sin(x)}{x}\right)^2 dx$$

Let $\epsilon > 0$ be given, then there is $A > 0$ large enough so that

$$\left| \int_0^{\infty} \left(\frac{\sin(x)}{x}\right)^2 dx - \int_0^{A} \left(\frac{\sin(x)}{x}\right)^2 dx \right| < \frac{\epsilon}{4}$$

Let $\delta = \frac{A}{M}$ for some large integer M and consider the partition $\{0, \delta, 2\delta, \cdots, A = M\delta\}$, so the Riemann sum of the integral is

$$\delta \sum_{n=1}^{M} \frac{\sin^2(n\delta)}{n^2\delta^2} = \sum_{n=1}^{M} \frac{\sin^2(n\delta)}{n^2\delta}$$

So there is M large enough so that

$$\left| \int_0^A \left(\frac{\sin(x)}{x} \right)^2 dx - \sum_{n=1}^{M} \frac{\sin^2(n\delta)}{n^2 \delta} \right| < \frac{\epsilon}{4}$$

From the def of a series, we can make M large enough so that

$$\left| \sum_{n=1}^{M} \frac{\sin^2(n\delta)}{n^2 \delta} - \left(\frac{\pi - \delta}{2} \right) \right| < \frac{\epsilon}{4}$$

Last but not least, make M large (so $\delta = \frac{A}{M}$ is small) so that

$$\left| \frac{\pi - \delta}{2} - \frac{\pi}{2} \right| < \frac{\epsilon}{4}$$

Combining all 4 pieces, we get the result