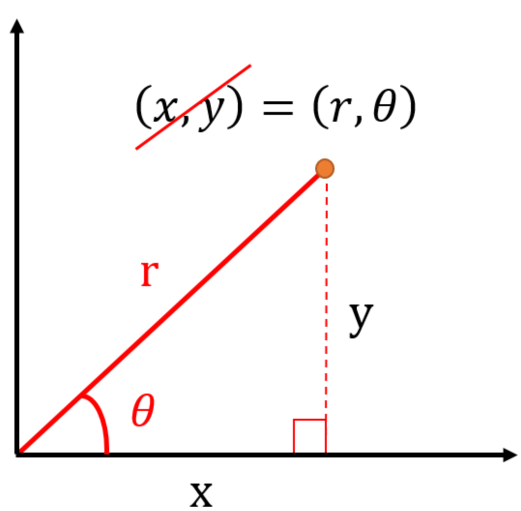
LECTURE 24: DOUBLE INTEGRALS IN POLAR COORDINATES

Today: It's getting chilly because we'll do double integrals in polar coordinates

1. Recap: Polar Coordinates

Main Idea: We can write (x, y) as (r, θ) where r is the distance between (x, y) and the origin, and θ is the angle between (x, y) and the $x-$axis:



Date: Friday, October 22, 2021.

Polar Coordinates:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

In addition, we often use the following

$$\begin{aligned} x^2 + y^2 &= r^2 \\ \tan^{-1}\left(\frac{y}{x}\right) &= \theta \end{aligned}$$

2. Integrals in Polar Coordinates

Let's use polar coordinates to evaluate some impossible integrals

Example 1:

Calculate the following integral, where D is the disk $x^2 + y^2 \leq 4$

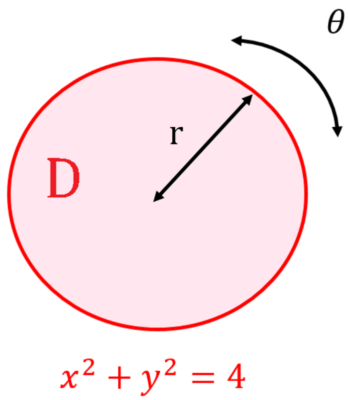
$$\int\int_D \left(x^2 + y^2\right)^{\frac{3}{2}} dxdy$$

Rule of Thumb: If you see $x^2 + y^2$ or disks, use polar coordinates

STEP 1: Write f in polar coordinates:

$$f(x,y) = \left(x^2 + y^2\right)^{\frac{3}{2}} = \left(r^2\right)^{\frac{3}{2}} = r^3 \Rightarrow f(r,\theta) = r^3$$

STEP 2: **Inequalities**: D is a disk of radius 2:



In terms of r and θ, this becomes:

$$\begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

STEP 3: Integrate

Polar Coordinate Formula:

$$\int\int_D f(x,y)dxdy = \int\int_D f(r,\theta)\, r\, drd\theta$$

⚠ ⚠ ⚠ Do NOT forget about the r !!! Think like a pirate, "Arrrrrrr dr dθ"

$$\begin{aligned}
&\int\int_D \left(\sqrt{x^2+y^2}\right)^3 dxdy \\
=&\int_0^{2\pi}\int_0^2 r^3 \, r dr d\theta \\
=&\int_0^{2\pi}\int_0^2 r^4 dr d\theta \\
=&\left(\int_0^2 r^4 dr\right)\left(\int_0^{2\pi} 1 d\theta\right) \\
=&(2\pi)\left[\frac{r^5}{5}\right]_0^2 \\
=&(2\pi)\frac{2^5}{5} \\
=&\frac{64\pi}{5}
\end{aligned}$$

Note: If you're curious why there is an r in $rdrd\theta$, check out the optional appendix at the end of the lecture notes.

3. Interlude: An important integral

Video: Trigonometric Integral

Before we continue, let me remind you of an important integral that we'll use many times in this course

Example 2:

Evaluate the following integral:

$$\int_0^{\frac{\pi}{4}} \cos^2(\theta) d\theta$$

It all starts with:

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ &= \cos^2(\theta) - \left(1 - \cos^2(\theta)\right)\\ &= 2\cos^2(\theta) - 1\end{aligned}$$

Now solve for $\cos^2(\theta)$:

$$\begin{aligned}2\cos^2(\theta) - 1 &= \cos(2\theta)\\ 2\cos^2(\theta) &= 1 + \cos(2\theta)\\ \cos^2(\theta) &= \frac{1}{2} + \frac{1}{2}\cos(2\theta)\end{aligned}$$

And finally integrate:

$$\begin{aligned}&\int_0^{\frac{\pi}{4}} \cos^2(\theta) d\theta\\ =&\int_0^{\frac{\pi}{4}} \frac{1}{2} + \frac{1}{2}\cos(2\theta) d\theta\\ =&\left[\frac{\theta}{2} + \frac{1}{2}\sin(2\theta)\left(\frac{1}{2}\right)\right]_0^{\frac{\pi}{4}} \quad (\text{think } u = 2\theta)\\ =&\frac{\pi}{8} + \frac{1}{4}\sin\left(\frac{\pi}{2}\right) - 0 - 0\\ =&\frac{\pi}{8} + \frac{1}{4} = \frac{\pi + 2}{8}\end{aligned}$$

Note: For $\int \sin^2(\theta)d\theta$, it's the same, except this time you have

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

4. MORE POLAR FUN

Video: Polar Integral

Example 3:

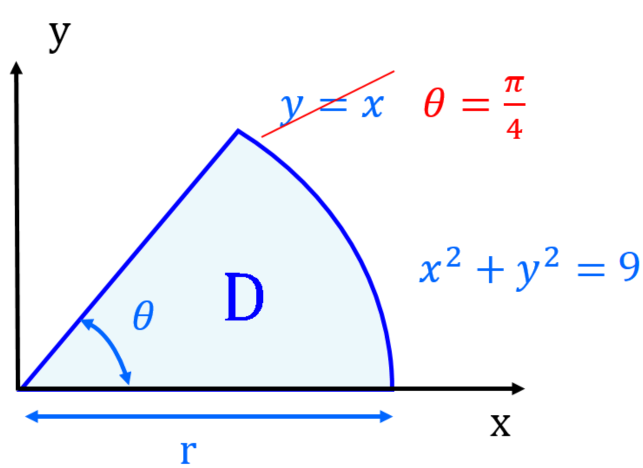
Evaluate the following, where D is the region in the first quadrant bounded by the circle $x^2 + y^2 = 9$, the $x-$axis, and the line $y = x$

$$\int\int_D x^2 dxdy$$

STEP 1: Function

$$f(x, y) = x^2 = (r\cos(\theta))^2 = r^2\cos^2(\theta)$$

STEP 2: Inequalities



The circle $x^2 + y^2 = 9$ has radius 3 and also $y = x$ corresponds to an angle of $\theta = \frac{\pi}{4}$ (45 degrees), hence D is:

$$\begin{cases} 0 \leq r \leq 3 \\ 0 \leq \theta \leq \dfrac{\pi}{4} \end{cases}$$

STEP 3: Integrate

$$\begin{aligned}
&\int\int_D f(x,y)dxdy \\
=&\int_0^{\frac{\pi}{4}}\int_0^3 r^2\cos^2(\theta)\, r drd\theta \\
=&\int_0^{\frac{\pi}{4}}\int_0^3 r^3\cos^2(\theta) drd\theta \\
=&\left(\int_0^3 r^3 dr\right)\left(\int_0^{\frac{\pi}{4}}\cos^2(\theta)d\theta\right) \\
=&\left[\frac{r^4}{4}\right]_0^3\left(\frac{\pi+2}{8}\right) \qquad \text{(See Example above)} \\
=&\frac{81}{4}\left(\frac{\pi+2}{8}\right) \\
=&\frac{81}{32}(\pi+2)
\end{aligned}$$

5. Volume between surfaces

Video: Volume of an ice cream cone

Since we worked so hard so far, let's treat ourselves with some ice cream and calculate the volume of an ice cream cone ☺

Example 4:

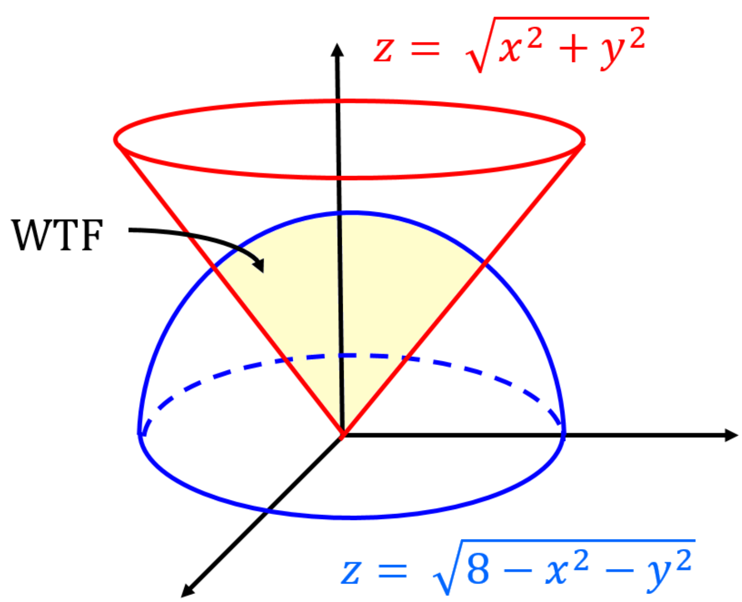
Find the volume of the region between

$$z = \sqrt{x^2 + y^2} \text{ and } z = \sqrt{8 - x^2 - y^2}$$

STEP 1: Picture

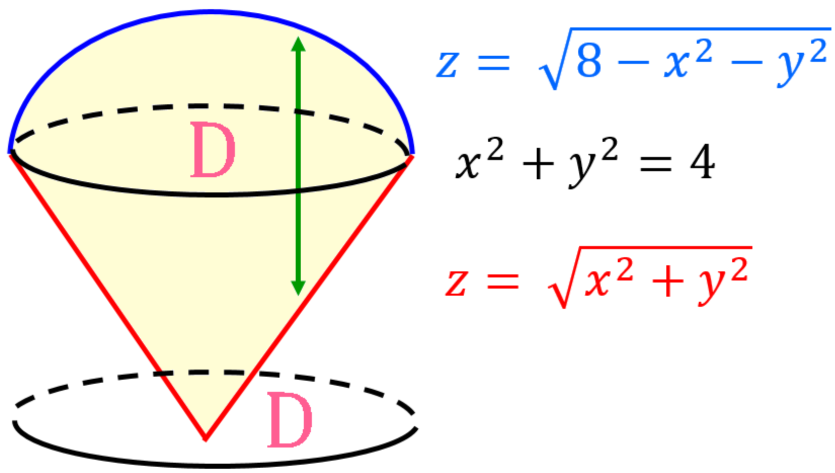
$z = \sqrt{x^2 + y^2} \Rightarrow z^2 = x^2 + y^2$ and $z \geq 0$, so an upper half-cone

$z = \sqrt{8 - (x^2 + y^2)} \Rightarrow z^2 = 8 - (x^2 + y^2) \Rightarrow x^2 + y^2 + z^2 = 8 \Rightarrow$ Upper Hemisphere



This looks messy, so let's redraw it

Re-draw:



STEP 2: Just for areas in calculus, here the volume is given by the integral of the bigger function minus the smaller function:

$$\text{Volume} = \int\int_D \text{Bigger} - \text{Smaller}\, dxdy = \int\int_D \sqrt{8 - x^2 - y^2} - \sqrt{x^2 + y^2} dxdy$$

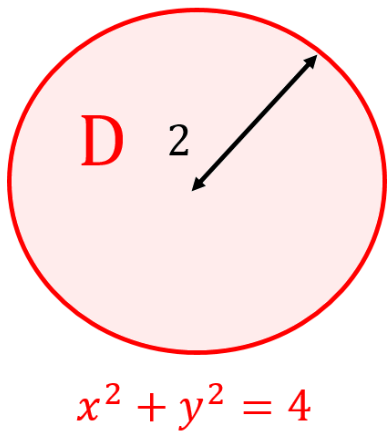
STEP 3: Find D

Here D is shadow under the solid, which is given precisely by the **intersection** of the two surfaces (this is typical)

Intersection:

$$\begin{aligned}\sqrt{8-x^2-y^2} =& \sqrt{x^2+y^2}\\ 8-x^2-y^2 =& x^2+y^2\\ 8 =& 2(x^2+y^2)\\ x^2+y^2 =& 4\end{aligned}$$

So $D =$ Disk of radius 2



STEP 4: Integrate

Function:

$$\sqrt{8-(x^2+y^2)} - \sqrt{x^2+y^2} = \sqrt{8-r^2} - \sqrt{r^2} = \sqrt{8-r^2} - r$$

Endpoints:

$$\begin{aligned}0 \leq r \leq 2\\ 0 \leq \theta \leq 2\pi\end{aligned}$$

Integral:

$$V = \int_0^{2\pi} \int_0^2 \left[\sqrt{8-r^2} - r\right] r \, dr d\theta = 2\pi \int_0^2 \left(\sqrt{8-r^2}\right) r - r^2 dr$$

$$\begin{aligned}
2\pi \int_0^2 \left(\sqrt{8-r^2}\right) r dr =& 2\pi \int_8^4 \sqrt{u} \left(-\frac{1}{2} du\right) \\
& \left(u = 8 - r^2, du = -2r dr \Rightarrow r dr = -\frac{1}{2} du, u(0) = 8, u(2) = 4\right) \\
=& 2\pi \int_4^8 \frac{1}{2} u^{\frac{1}{2}} du \\
=& 2\pi \left[\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) u^{\frac{3}{2}}\right]_4^8 \\
=& 2\pi \left[\left(\frac{1}{3}\right) u\sqrt{u}\right]_4^8 \\
=& \frac{2\pi}{3}\left(8\sqrt{8} - 4\sqrt{4}\right) \\
=& \frac{2\pi}{3}\left(8(2\sqrt{2}) - 4(2)\right) \\
=& \frac{2\pi}{3}\left(16\sqrt{2} - 8\right) \\
=& \left(\frac{32\pi}{3}\right)\sqrt{2} - \left(\frac{16\pi}{3}\right)
\end{aligned}$$

$$2\pi \int_0^2 -r^2 dr = 2\pi \left[-\frac{1}{3} r^3\right]_0^2 = -\frac{2\pi}{3}(8-0) = -\frac{16\pi}{3}$$

$$V = \left(\frac{32\pi}{3}\right)\sqrt{2} - \frac{16\pi}{3} - \frac{16\pi}{3} = \left(\frac{32\pi}{3}\right)\sqrt{2} - \frac{32\pi}{3} = \frac{32\pi}{3}\left(\sqrt{2} - 1\right)$$

6. Integral over a ring

Video: Integral over a ring

As Beyoncé once said: If you like it then you shoulda put a *ring* on it

Example 5: (extra practice)

Find the average value of $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ over the ring D between the circles $x^2+y^2 = a^2$ and $x^2+y^2 = b^2$ (where $0 < a < b$)

Recall:

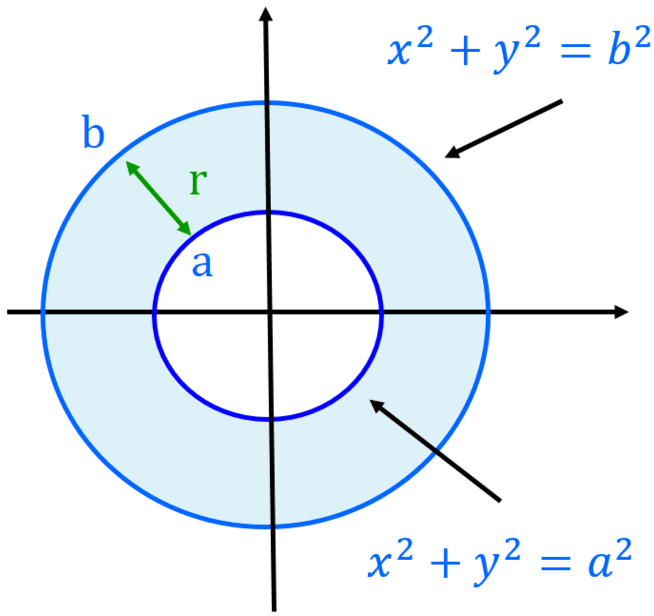
The average value of $f(x,y)$ over D is

$$\frac{\int\int_D f(x,y)dxdy}{\text{Area of } D}$$

STEP 1: Function:

$$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right) = \theta \qquad \text{(Much easier)}$$

STEP 2: Find D



Inequalities:

$$a \leq r \leq b$$
$$0 \leq \theta \leq 2\pi$$

STEP 3: Integrate:

$$\begin{aligned}
&\int\int_D f(x,y)dxdy \\
&= \int_0^{2\pi}\int_a^b \theta r \, dr d\theta \\
&= \left(\int_0^{2\pi} \theta d\theta\right)\left(\int_a^b r dr\right) \\
&= \left[\frac{\theta^2}{2}\right]_0^{2\pi}\left[\frac{r^2}{2}\right]_a^b
\end{aligned}$$

$$
\begin{aligned}
=&\frac{(2\pi)^2}{2}\left(\frac{b^2}{2}-\frac{a^2}{2}\right)\\
=&\frac{4\pi^2}{4}\left(b^2-a^2\right)\\
=&\pi^2\left(b^2-a^2\right)
\end{aligned}
$$

STEP 4: Average

Since D is the difference between two disks, so the area of D is:

$$\text{Area }(D)=\pi b^2-\pi a^2=\pi\left(b^2-a^2\right)$$

Therefore the average is given by:

$$\frac{\text{Integral}}{\text{Area}}=\frac{\pi^2\left(b^2-a^2\right)}{\pi\left(b^2-a^2\right)}=\frac{\pi^2}{\pi}=\pi$$

Note: Notice how this answer is independent of the a and b, WOW!!!

7. More Practice

Example 6: (extra practice)

$$\int_0^2\int_0^{\sqrt{2x-x^2}}9\sqrt{x^2+y^2}\,dydx$$

STEP 1: Function:

$$f(x,y)=9r$$

STEP 2: Find D

$$
\begin{aligned}
0\leq&y\leq\sqrt{2x-x^2}\\
0\leq&x\leq 2
\end{aligned}
$$

$$
\begin{aligned}
y = \sqrt{2x - x^2} \Rightarrow & y^2 = 2x - x^2 \\
\Rightarrow & x^2 - 2x + y^2 = 0 \\
\Rightarrow & (x-1)^2 - 1 + y^2 = 0 \\
\Rightarrow & (x-1)^2 + y^2 = 1
\end{aligned}
$$

So D is an upper half circle centered at $(1, 0)$ and radius 1

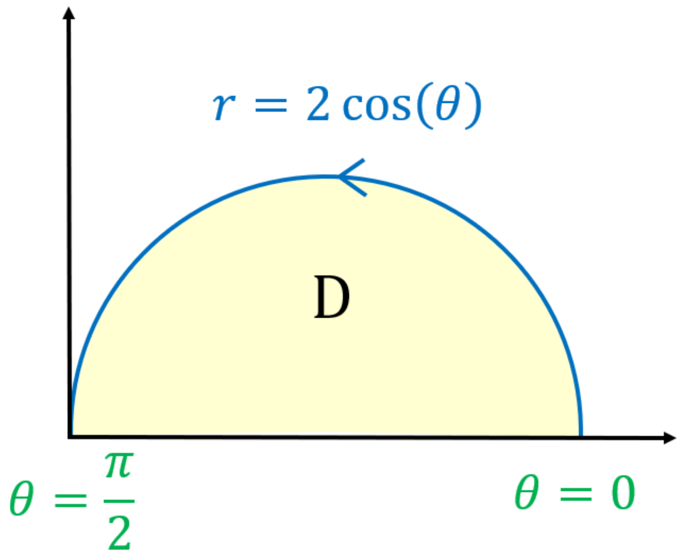
STEP 3: Inequalities

In terms of polar coordinates, notice that

$$
\begin{aligned}
y^2 =& 2x - x^2 \\
(r\sin(\theta))^2 =& 2r\cos(\theta) - (r\cos(\theta))^2 \\
r^2\sin^2(\theta) =& 2r\cos(\theta) - r^2\cos^2(\theta) \\
r^2\cos^2(\theta) + r^2\sin^2(\theta) =& 2r\cos(\theta) \\
r^2 =& 2r\cos(\theta) \\
r =& 2\cos(\theta)
\end{aligned}
$$

Moreover, if $\theta = 0$, we get $r = 2$ (right endpoint of circle) and if $\theta = \frac{\pi}{2}$, we get $r = 0$ (left endpoint of circle), hence θ goes from 0 to $\frac{\pi}{2}$ (and not 0 to π, as one might at first expect), therefore our inequalities are:

$$
\begin{aligned}
0 \leq & r \leq 2\cos(\theta) \\
0 \leq & \theta \leq \frac{\pi}{2}
\end{aligned}
$$



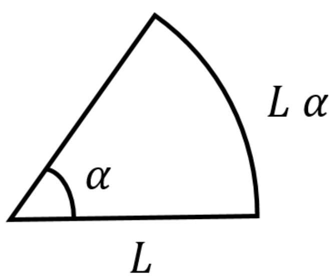
STEP 4: Integrate:

$$\begin{aligned}
&\int\int_D f(x,y)dxdy \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} 9r{\color{red}r}\, drd\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} 9r^2\, drd\theta \\
&= \int_0^{\frac{\pi}{2}} \left[3r^3\right]_0^{2\cos(\theta)} d\theta \\
&= \int_0^{\frac{\pi}{2}} 3\left(2\cos(\theta)\right)^3 d\theta \\
&= \int_0^{\frac{\pi}{2}} (3)\, 8\cos^3(\theta)d\theta \\
&= \int_0^{\frac{\pi}{2}} 24\cos^2(\theta)\cos(\theta)d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{2}} 24\left(1-\sin^2(\theta)\right)\cos(\theta)d\theta \\
&\quad \left(u=\cos(\theta), du=-\sin(\theta)d\theta, u(0)=1, u\left(\frac{\pi}{2}\right)=0\right) \\
&= \int_1^0 24\left(1-u^2\right)(-du) \\
&= \int_0^1 24-24u^2du \\
&= \left[24u^2-8u^4\right]_0^1 \\
&=24-8-0+0 \\
&=16
\end{aligned}
$$

8. Optional Appendix: $rdrd\theta$

Where does the r in $rdrd\theta$ come from? Here is a rough explanation.



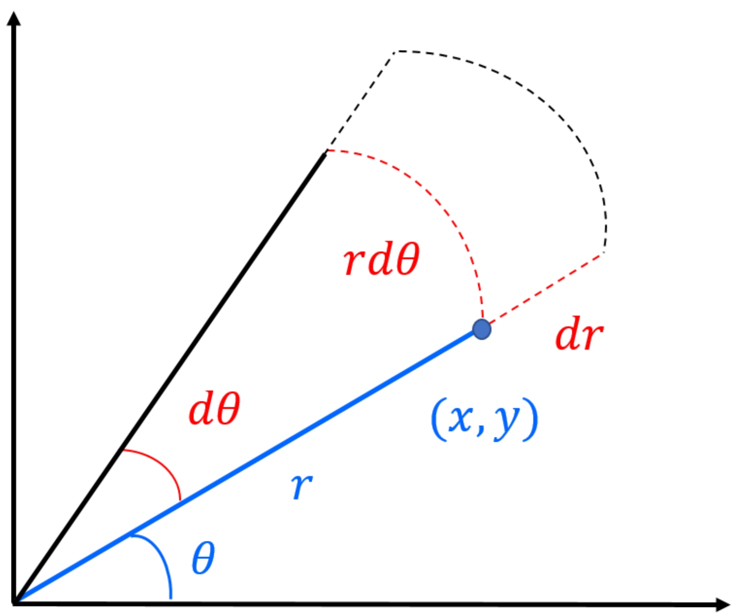
Recall:

The length of an arc of radius L and angle α is $L\alpha$

Why? An angle of 2π corresponds to a length of $2\pi L$, so an angle of α corresponds to a length of αL

Suppose you start at a point (x, y)

Thinking in terms of polar, change the radius by a tiny amount dr and the angle by a tiny amount $d\theta$. Then you get the following wedge:



The radius of the (inside) wedge is $L = r$ and the angle is $\alpha = d\theta$, so by the above formula the length of the (inside) wedge is $rd\theta$.

The thickness of the wedge is dr

Thinking of the wedge as a rectangle, the area becomes approximately

$$\text{Area} \approx \text{Length} \times \text{Thickness} = rd\theta \times dr = r\,drd\theta$$

Which is where the $rdrd\theta$ comes from

Of course, this needs a bit more justification: Why is this the same as $dxdy$? And why can you just say that the wedge is like a rectangle (which it technically isn't), but at least this explains roughly where the r comes from.