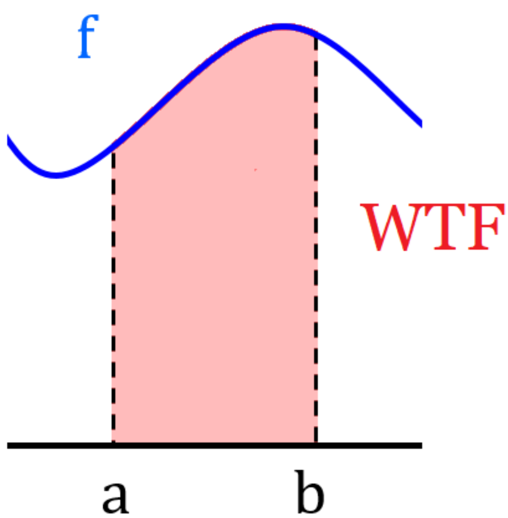
LECTURE 24: RIEMANN INTEGRAL (I)

Welcome to the final chapter of our course! This chapter will be *integral* to our analysis adventure, because it's all about integration!

1. The Darboux Integral

Video: Darboux Integral

Goal: Find the area under the graph of f on $[a, b]$

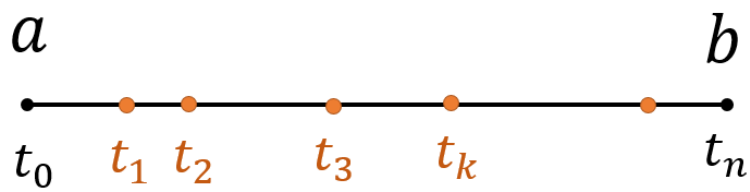


Note: In this chapter, f is bounded, but not necessarily continuous

Note: Here we'll take a slightly different approach from what you learned in Calculus; this one is more suitable for theoretical purposes.

Date: Thursday, November 18, 2021.

STEP 1: Divide $[a, b]$ into sub-pieces



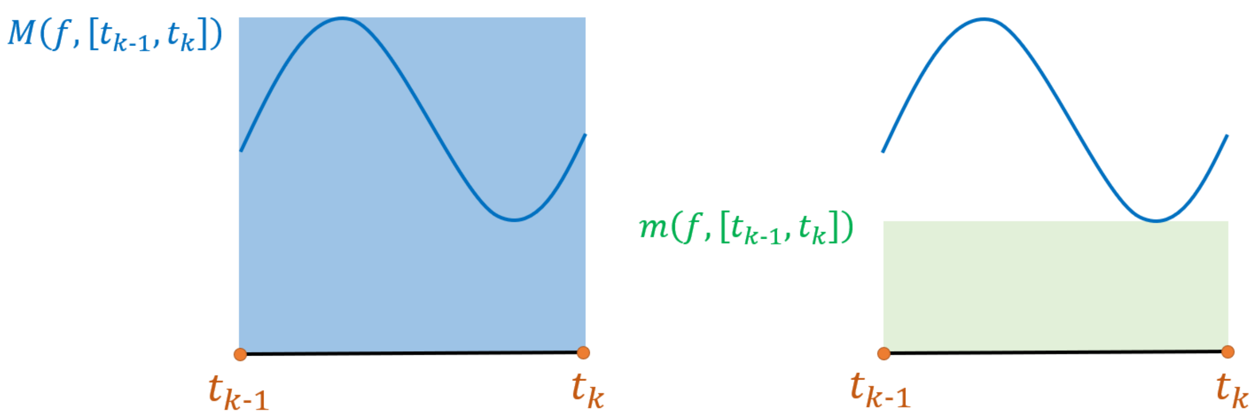
Definition:

A **partition** of $[a, b]$ is a collection of points of the form

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

Warning: Here the points t_k are not evenly spaced! This makes it somewhat more flexible.

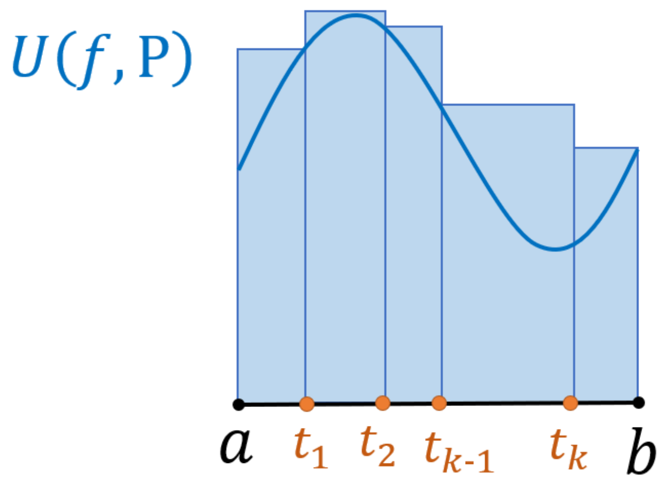
STEP 2: On each sub-piece $[t_{k-1}, t_k]$, consider the rectangle with height the biggest value of f and the smallest value of f

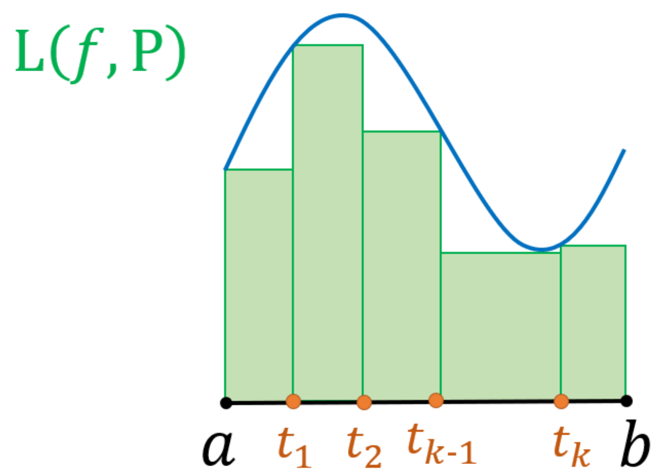


Definition:

$$M(f,[t_{k-1},t_k]) = \sup\{\ f(x)\mid x \text{ in } [t_{k-1},t_k]\ \}$$
$$m(f,[t_{k-1},t_k]) = \inf\{\ f(x)\mid x \text{ in } [t_{k-1},t_k]\ \}$$

STEP 3: Sum up the areas of all the rectangles



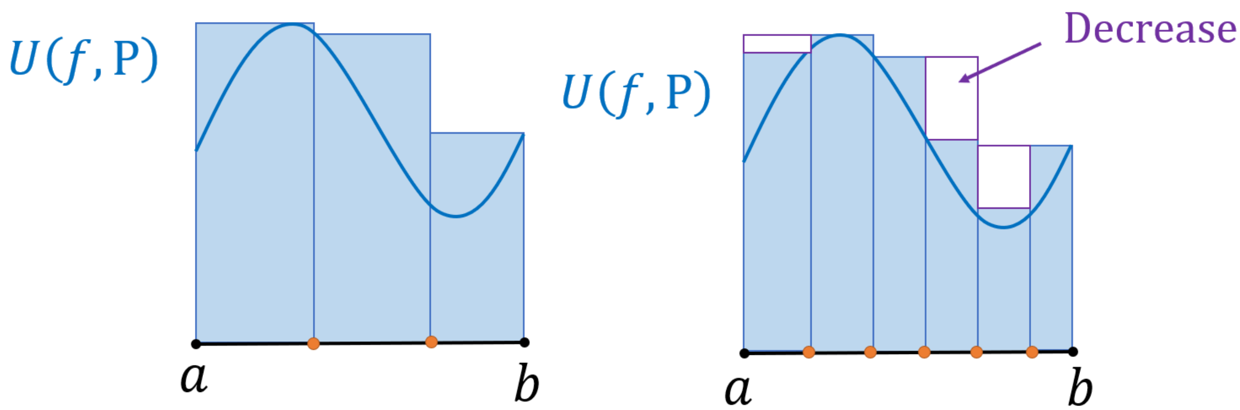


Definition:

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) \text{ (Upper Sum)}$$
$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1}) \text{ (Lower Sum)}$$

U is an overestimate and L is an underestimate; the actual area lies in between.

Important Observation: If you increase the number of rectangles, then U decreases, as in the following picture, where we use 3 vs 6 rectangles. Similarly, L increases



Because of this, it makes sense to consider:

Definition: (Upper/Lower Darboux Integral)

$$U(f) = \inf \{ \, U(f,P) \mid P \text{ is a partition of } [a,b] \, \} \quad \text{(Upper)}$$
$$L(f) = \sup \{ \, L(f,P) \mid P \text{ is a partition of } [a,b] \, \} \quad \text{(Lower)}$$

Even though it's true that $L(f) \leq U(f)$, it is not always true that $L(f) = U(f)$. That said:

Definition:

We say f is **integrable** on $[a,b]$ if $L(f) = U(f)$. In that case, the **Darboux integral** of f is

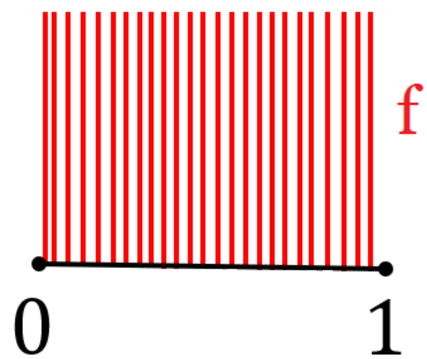
$$\int_a^b f(x)dx = L(f) = U(f)$$

2. Examples

Non-Example: 1

Consider the following function on $[0,1]$:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$



Then $M(f,[t_{k-1},t_k]) = 1$ but $m(f,[t_{k-1},t_k]) = 0$, so

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} t_k - t_{k-1} \\ &= (t_1 - t_0) + (t_2 - t_1) + \cdots + (t_n - t_{n-1}) \\ &= t_n - t_0 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Since $U(f, P) = 1$ for all P, $U(f) = \inf \{ U(f, P) \mid P \} = 1$

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} 0(t_k - t_{k-1}) = 0$$

Therefore $L(f) = 0$

Since $L(f) \neq U(f)$, f is not Darboux integrable

Note: This doesn't mean that f is bad, it just means that our theory of integration sucks! There is a more powerful theory called the Lebesgue integral, which takes care precisely of functions like those

Example: 2

Consider $f(x) = x^2$ on $[0, 1]$

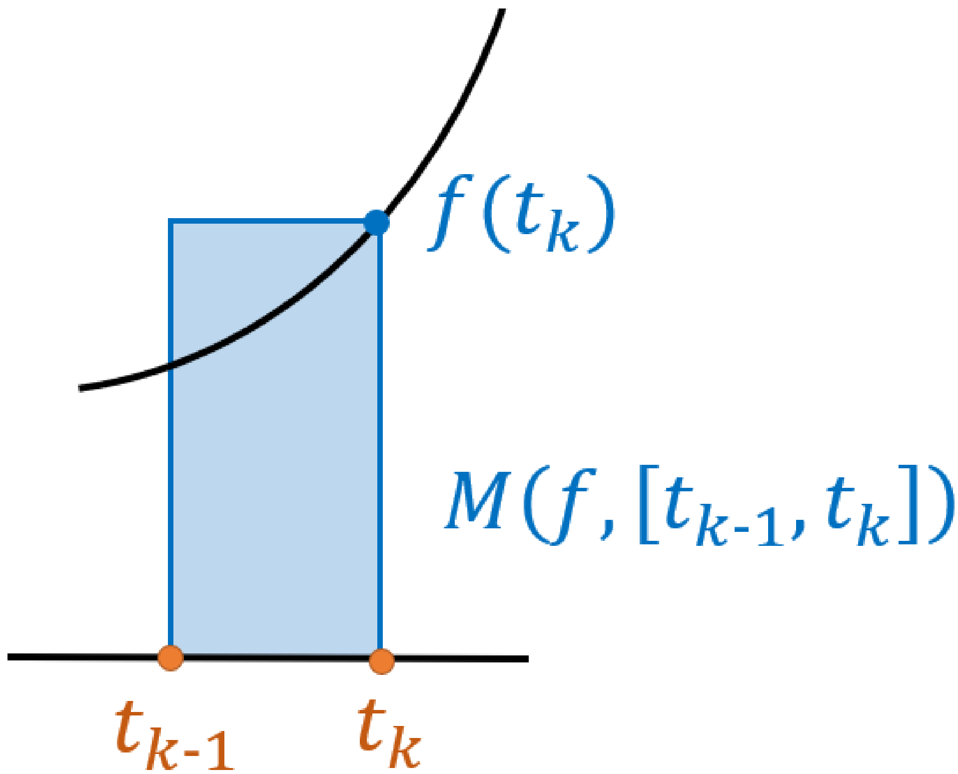
STEP 1: Partition

$$P = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$$

STEP 2: $U(f, P)$

Observation: Since x^2 is increasing, notice that:

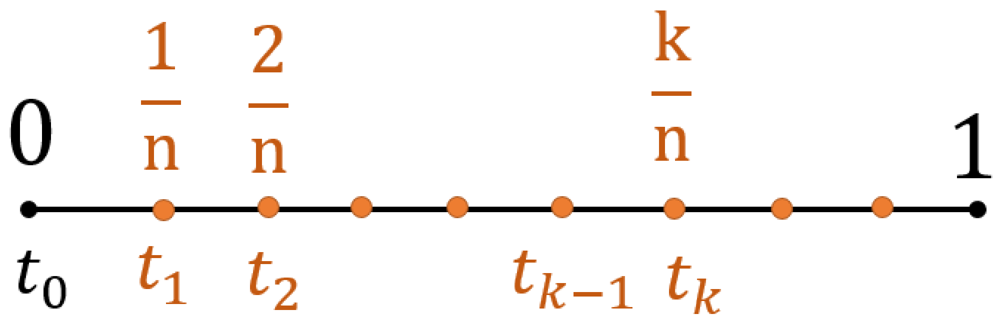
$$M(f, [t_{k-1}, t_k]) = f(t_k) = (t_k)^2 \quad \text{(Right Endpoint)}$$



$$U(f,P)=\sum_{k=1}^{n} M(f,[t_{k-1},t_k])\,(t_k-t_{k-1})=\sum_{k=1}^{n}(t_k)^2\,(t_k-t_{k-1})$$

STEP 3: $U(f)$

Given n, let P be the evenly spaced Calculus partition with $t_k=\frac{k}{n}$:



In that case $t_k - t_{k-1} = \frac{1}{n}$ and

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \left(\frac{1}{n}\right) \\ &= \sum_{k=1}^{n} \frac{k^2}{n^3} \\ &= \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\ &= \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) \quad \text{(Will be given)} \\ &= \frac{(n+1)(2n+1)}{6n^2} \end{aligned}$$

Upshot: Since $U(f)$ is the inf over all partitions, we must have

$$U(f) \leq U(f,P) = \frac{(n+1)(2n+1)}{6n^2}$$

Therefore, taking the limit as $n \to \infty$ of the right hand side[1], we get $U(f) \leq \frac{2}{6} = \frac{1}{3}$, and so $U(f) \leq \frac{1}{3}$

[1] Here we used that if $a \leq s_n$, then so is $a \leq s$, where s is the limit of s_n

STEP 4: $L(f)$

This is similar to the above, except that here $m(f,[t_{k-1},t_k]) = (t_{k-1})^2$ (Left endpoint), and so, using sup we get $L(f) \geq \frac{1}{3}$.

Since $U(f) \leq \frac{1}{3} \leq L(f)$ and because $L(f) \leq U(f)$, we get $L(f) = U(f) = \frac{1}{3}$. Hence $f(x) = x^2$ is Darboux integrable and $\int_0^1 x^2 dx = \frac{1}{3}$.

3. $L(f) \leq U(f)$

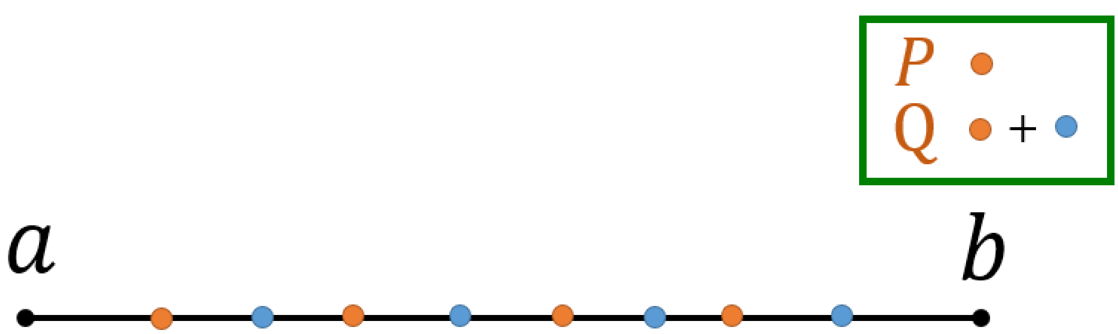
For the rest of today, our goal is to show that $L(f) \leq U(f)$. Although intuitively obvious, mathematically it's not because L is the sup of inf, whereas U is the inf of sup (like a minimax problem)

First, as remarked above, let's show that increasing the number of rectangles causes U to decrease:

Lemma 1

If P and Q are partitions of $[a,b]$ with $P \subseteq Q$, then $U(f,Q) \leq U(f,P)$

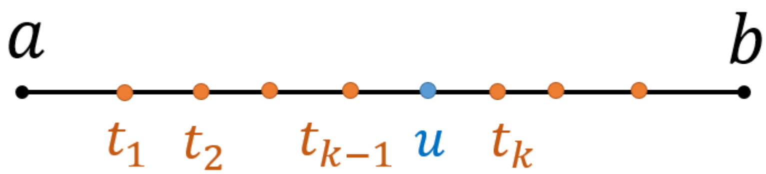
Note: Here Q is called a **refinement** of P



Proof: For simplicity, assume Q has one more point than P. Otherwise, just repeat this proof for the other points (or use induction)

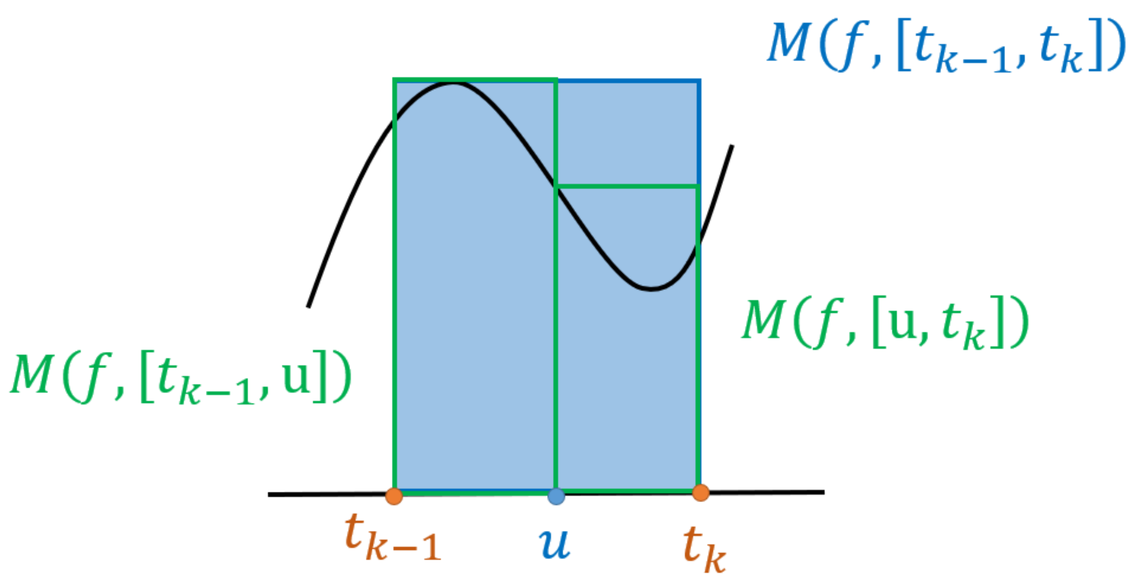
Suppose $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ as usual, and

$$Q = \{a = t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b\}$$



Q is basically P, but you insert an extra point u between t_{k-1} and t_k

Then the main thing to notice is simply that the rectangle with base $[t_{k-1}, t_k]$ is larger than the rectangles with bases $[t_{k-1}, u]$ and $[u, t_k]$, as in the following picture:



$$M(f,[t_{k-1},t_k])(t_k-t_{k-1}) \geq M(f,[t_{k-1},u])(u-t_{k-1})+M(f,[u,t_k])(t_k-u)$$

$$\begin{aligned}
&U(f,P)\\
=&M(f,[t_0,t_1])(t_1-t_0)+\cdots+M(f,[t_{k-1},t_k])(t_k-t_{k-1})+\cdots\\
\geq&M(f,[t_0,t_1])(t_1-t_0)+\cdots+M(f,[t_{k-1},u])(u-t_{k-1})+M(f,[u,t_k])(t_k-u)+\cdots\\
=&U(f,Q)
\end{aligned}$$

Hence $U(f,P) \geq U(f,Q)$ □

Note: Similarly, if $P \subseteq Q$, we have $L(f,P) \leq L(f,Q)$, so we get

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$

So the finer the partition, the better the approximation $L(f,Q)$ and $U(f,Q)$. This should give you Pre-Ratio Test vibes ☺

Next we'll show that for *all* partitions, the upper sums are bigger than the lower sums:

Lemma 2

For *any* partitions P and Q, we have $L(f,P) \leq U(f,Q)$

Proof: Notice $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$, and so by Lemma 1:

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q) \quad \square$$

Now we're finally ready to prove our main result:

Theorem:

$$L(f) \leq U(f)$$

Proof: For any partition P and Q, from Lemma 2, we have

$$L(f, P) \leq U(f, Q)$$

The left hand side is indep. of Q, so taking the inf over Q, we get

$$L(f, P) \leq \inf \{ U(f, Q) \mid Q \text{ is a partition of } [a, b] \} =: U(f)$$

Therefore for every P, we have

$$L(f, P) \leq U(f)$$

Now taking the sup over all partitions P we get

$$\sup \{ L(f, P) \mid P \text{ is a partition of } [a, b] \} \leq U(f)$$

That is $L(f) \leq U(f)$ $\square$