LECTURE 25: TRIPLE INTEGRALS (I)

1. Gaussian Integral

Video: Gaussian Integral

Warning: This is the most exciting example of the course! **Nothing** in your life will be as exciting as this!

Recall:

e^{-x^2} does not have an antiderivative, and yet...

Example 1:

Calculate $\int_{-\infty}^{\infty} e^{-x^2} dx$

Using multivariable calculus, we're going to do the impossible!!!

Note: This is sometimes called the Gaussian integral, due to Karl Friedrich Gauss

Date: Monday, October 25, 2021.

STEP 1: **Trick:** Consider:

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-y^2} dy > 0$$

(It doesn't matter which variable x or y we're using; potato potahto)

STEP 2: **Multiply:**

$$\begin{aligned}
I^2 =& (I)(I) \\
=& \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) \\
=& \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) e^{-y^2} dy \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx dy \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \\
=& \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r dr d\theta \\
=& 2\pi \int_{0}^{\infty} r e^{-r^2} dr \\
=& 2\pi \left[\left(-\frac{1}{2}\right) e^{-r^2}\right]_{r=0}^{r=\infty} \qquad \text{(u-sub: } u = -r^2\text{)} \\
=& 2\pi \left(-\frac{1}{2} e^{-\infty} + \frac{1}{2} e^{0}\right) \\
=& 2\pi \left(0 + \frac{1}{2}\right) \\
=& \pi
\end{aligned}$$

STEP 3: Therefore $I^2 = \pi$ and hence, since $I > 0$, we get $I = \sqrt{\pi}$, and the answer is:

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Note: You can use the same technique to evaluate $\int_{-\infty}^{\infty} \sin(x^2)dx$, check out this video if you're interested:

Video: Integral $\sin(x^2)$ from $-\infty$ to ∞

Check out this playlist for 12 ways of evaluating the Gaussian integral:

Playlist: Gaussian Integral 12 Ways

2. Basic Example

Video: Triple Integrals

Welcome to triple integrals; It's triple the fun, but quadruple the pain!

Fortunately, the process is basically the same as doing double integrals.

Example 2:

Calculate the following integral, where E is the tetrahedron in the first octant bounded by $6x + 3y + z = 12$

$$\int\int\int_E 6z \, dxdydz$$

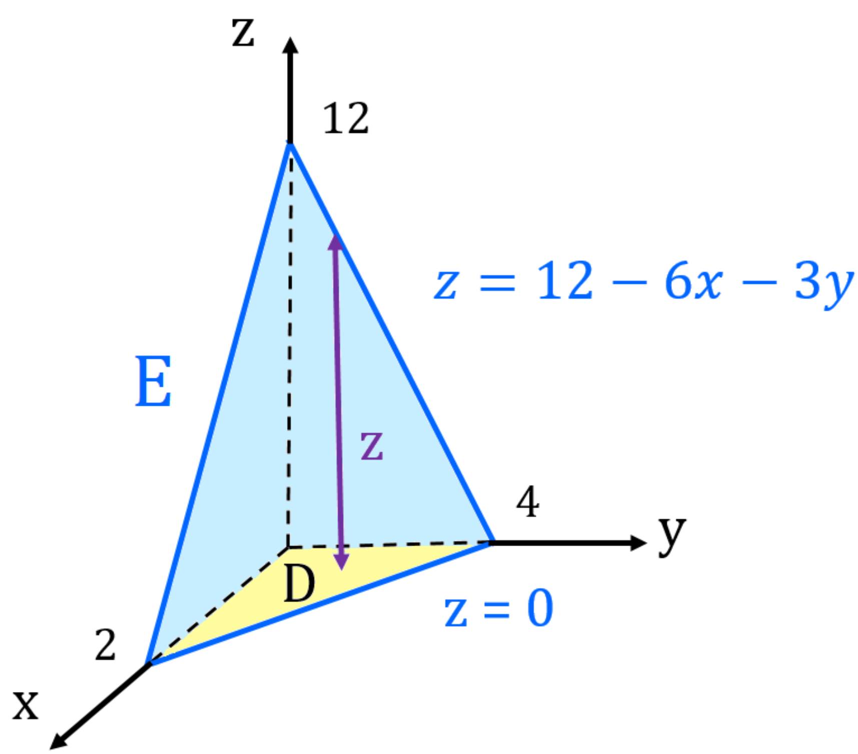
STEP 1: Picture

To draw the picture, again find the intercepts:

$z-$**intercept:** $6(0) + 3(0) + z = 12 \Rightarrow z = 12$

$y-$**intercept:** $6(0) + 3y + 0 = 12 \Rightarrow y = 4$

$x-$**intercept:** $6x + 3(0) + 0 = 12 \Rightarrow x = 2$

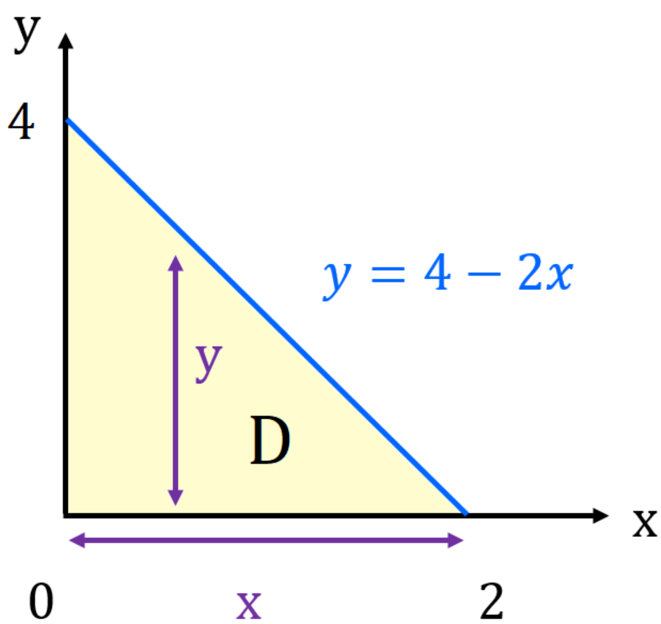


STEP 2: Inequalities

The main difference is that before we had $z = $ Big $-$ Small , but now we have inequalities

$$\text{Small } \leq z \leq \text{ Big}$$
$$0 \leq z \leq 12 - 6x - 3y$$

STEP 3: Find D



$$\text{Small } \leq y \leq \text{ Big}$$
$$0 \leq y \leq 4 - 2x$$

$$0 \leq x \leq 2$$

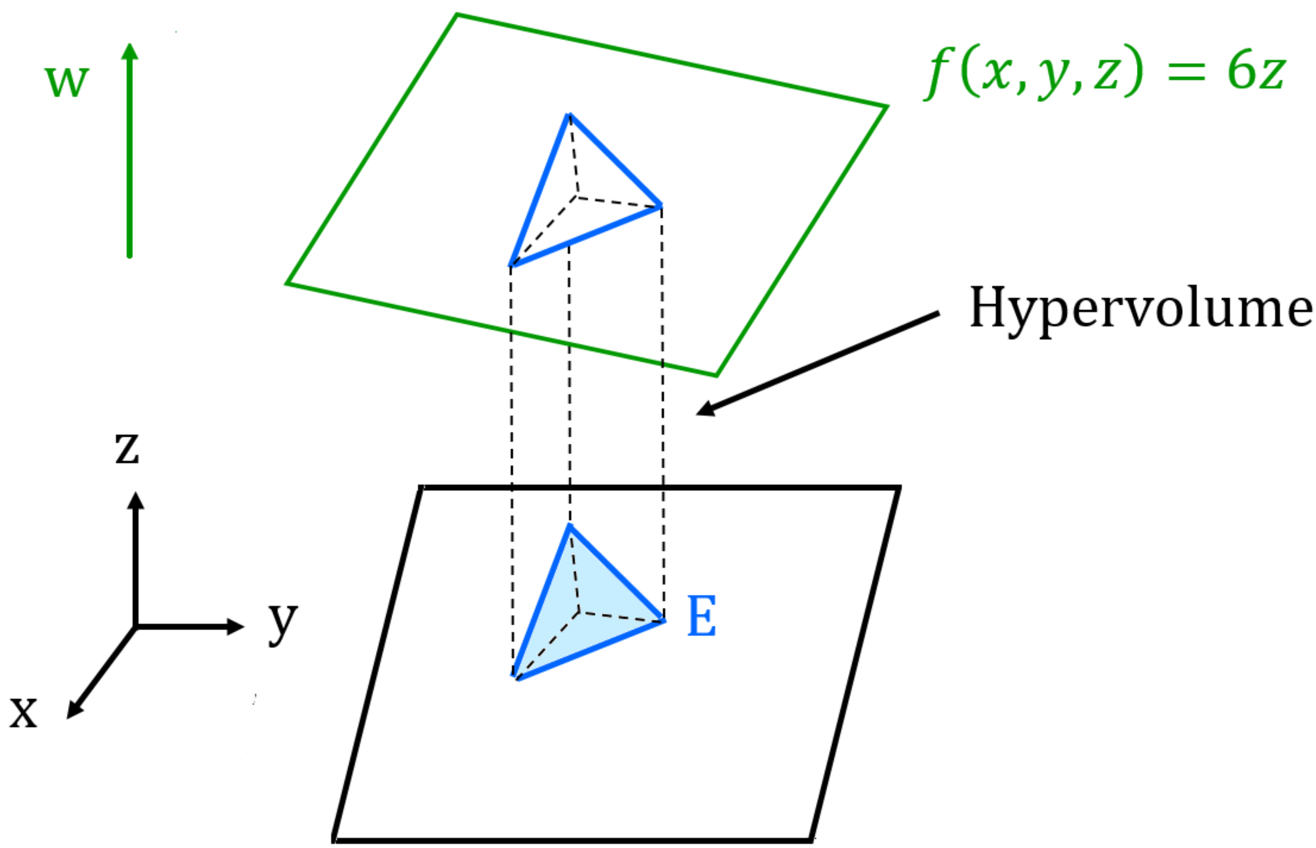
In summary, our inequalities are

$$\begin{cases} 0 \leq z \leq 12 - 6x - 3y \\ 0 \leq y \leq 4 - 2x \\ 0 \leq x \leq 2 \end{cases}$$

STEP 4: Integrate

$$
\begin{aligned}
&\int\int\int_E 6z \, dxdydz \\
=&\int_0^2 \int_0^{4-2x} \int_0^{12-6x-3y} 6z \, dzdydx \\
=&\int_0^2 \int_0^{4-2x} \left[3z^2\right]_{z=0}^{z=12-6x-3y} dydx \\
=&\int_0^2 \int_0^{4-2x} 3\left(12-6x-3y\right)^2 dydx \qquad (\textit{Double} \text{ Integral}) \\
=&\int_0^2 \left[\left(12-6x-3y\right)^3 \left(\frac{1}{-3}\right)\right]_{y=0}^{y=4-2x} dx \qquad (u = 12-6x-3y) \\
=&\int_0^2 \left(-\frac{1}{3}\right)\left(12-6x-3(4-2x)\right)^3 + \frac{1}{3}\left(12-6x-0\right)^3 dx \\
=&\frac{1}{3}\int_0^2 \left(12-6x\right)^3 dx \\
=&\left(\frac{1}{3}\right)\left[\left(\frac{1}{4}\right)\left(12-6x\right)^4\left(\frac{1}{-6}\right)\right]_0^2 \qquad (u = 12-6x) \\
=&-\frac{1}{72}\left(12-6(2)\right)^4 + \frac{1}{72}\left(12\right)^4 \\
=&\frac{(12)^4}{72} \\
=&288
\end{aligned}
$$

⚠ In general, a triple integral does **NOT** calculate a volume. It calculates the **hyper**volume (4D volume) under the function $f(x, y, z) = 6z$ and over the 3D solid E. Think of your base as E and the height $6z$ in the 4th dimension.



3. Mass

That said, there is a nice physical interpretation of triple integrals:

Fact:

If E is a solid with density $f(x, y, z)$, then the **mass** of E is

$$\int\int\int_E f(x, y, z)\, dxdydz$$

Example 3:

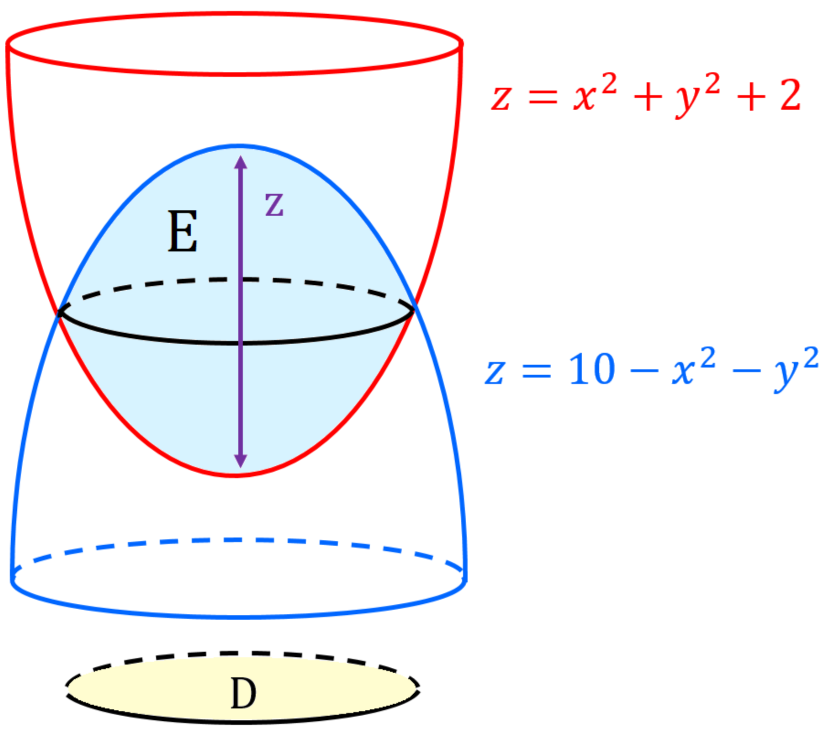
Find the mass of the solid E with density $f(x, y, z) = 2z$ between the paraboloids

$$z = x^2 + y^2 + 2 \text{ and } z = 10 - x^2 - y^2$$

In other words, we would like to calculate

$$\int\int\int_E 2z \, dxdydz$$

STEP 1: Picture:



STEP 2: Inequalities

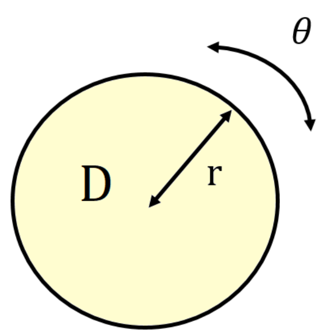
$$\begin{aligned} \text{Small } \leq &z \leq \text{ Big} \\ x^2 + y^2 + 2 \leq &z \leq 10 - x^2 - y^2 \end{aligned}$$

STEP 3: Find D

As usual, D is given by the intersection of the two surfaces.

Intersection:

$$\begin{aligned} 2 + x^2 + y^2 =& 10 - x^2 - y^2 \\ 2x^2 + 2y^2 =& 10 - 2 = 8 \\ x^2 + y^2 =& 4 \end{aligned}$$



Therefore D is a disk of radius 2, and hence

$$\begin{aligned} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{aligned}$$

STEP 4: Integrate

$$\begin{aligned}
\text{Mass} &= \int\int_E 2z\, dxdydz \\
&= \int\int_D \int_{x^2+y^2+2}^{10-x^2-y^2} 2z\, dzdxdy \\
&= \int\int_D \left[z^2\right]_{z=x^2+y^2+2}^{z=10-x^2-y^2}\, dxdy \\
&= \int\int_D \left(10 - x^2 - y^2\right)^2 - \left(x^2 + y^2 + 2\right)^2\, dxdy \\
&= \int_0^{2\pi}\int_0^2 \left[\left(10 - r^2\right)^2 - \left(r^2 + 2\right)^2\right] r drd\theta \\
&= 2\pi \int_0^2 \left[100 - 20r^2 + r^4 - r^4 - 4r^2 - 4\right] rdr \\
&= 2\pi \int_0^2 \left[96 - 24r^2\right] rdr \\
&= 2\pi \int_0^2 96r - 24r^3 dr \\
&= 2\pi \left[48r^2 - 6r^4\right]_0^2 \\
&= 2\pi \left(48(4) - 6(16)\right) \\
&= 2\pi(96) \\
&= 192\pi
\end{aligned}$$