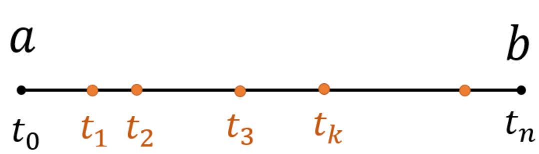
LECTURE 25: RIEMANN (II) + PROPERTIES (I)

1. The Riemann Integral

You may wonder: In what way is Darboux Integration similar or different from what you learned in Calculus about Riemann Integrals? Even though they look different, it turns out they are two different sides of the same coin! For this, let me remind you how Riemann integrals work:

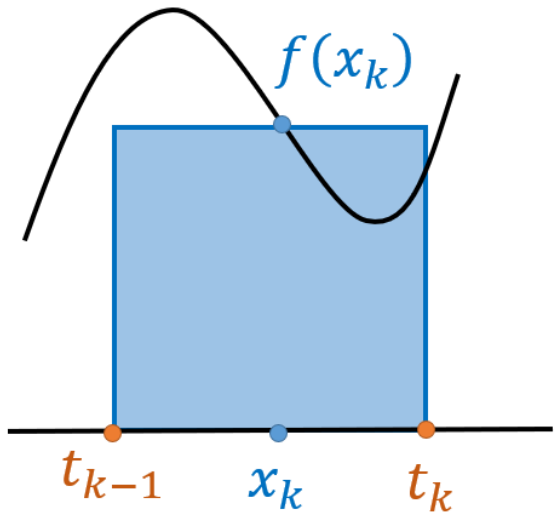
STEP 1: Divide $[a, b]$ into sub-pieces, so we define partitions as before

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

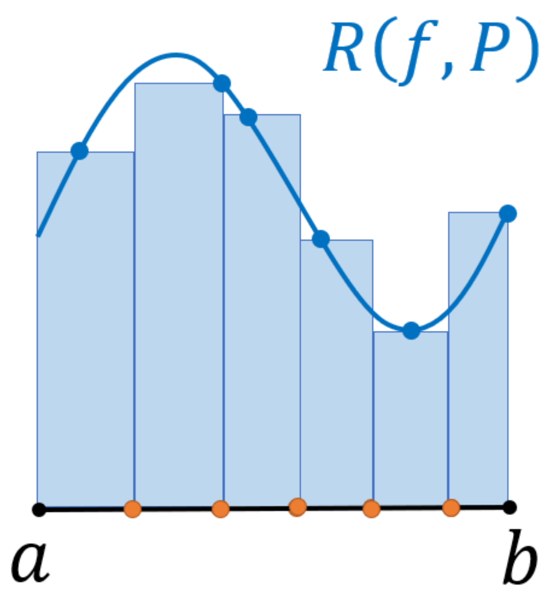


STEP 2: On each sub-piece $[t_{k-1}, t_k]$, choose a *random* point x_k and consider the rectangle with height $f(x_k)$

Date: Tuesday, November 23, 2021.



STEP 3: Sum up the areas of all the rectangles



Definition:

A **Riemann sum** of f associated with the partition P is a sum of the form

$$R(f, P) = \sum_{k=1}^{n} f(x_k)\,(t_k - t_{k-1})$$

Where x_k is a random point in $[t_{k-1}, t_k]$

Here $R(f, P)$ depends not only on P but also on the choice of x_k, think "random" sum

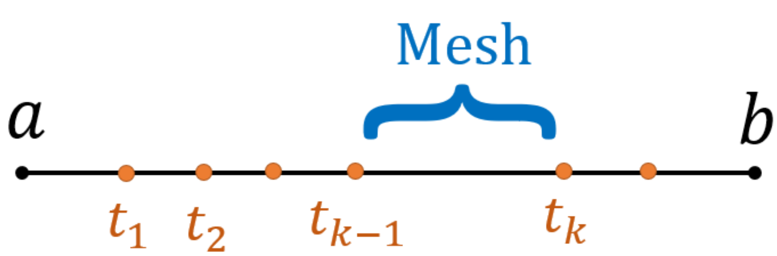
STEP 4:

In calculus, the next step was to take the limit as $n \to \infty$. Here we have to be more careful because the t_k are not evenly spaced.

Definition:

The **mesh** of P is the length of the largest sub-piece, that is

$$|P| = \max\{|t_k - t_{k-1}|\,, k = 1, \cdots, n\}$$



Intuitively we want to say that:

$$\lim_{|P| \to 0} R(f, P) = \int_a^b f(x)dx$$

That is, as the partitions P get very fine, *every* Riemann sum converges to some number called $\int_a^b f(x)dx$. To make this more precise, we have to use $\epsilon - \delta$

Definition:

f is **Riemann Integrable** on $[a, b]$ if there is a number $\int_a^b f(x)dx$ such that:

For all $\epsilon > 0$ there is a $\delta > 0$ such that for every partition P with mesh $< \delta$, and every Riemann sum $R(f, P)$, we have

$$\left| R(f, P) - \int_a^b f(x)dx \right| < \epsilon$$

If it exists, that number $\int_a^b f(x)dx$ is the **Riemann Integral** of f.

Notice how this is different from Darboux Integration. In Darboux, we took the upper sum $U(f)$ and lower sum $L(f)$ said that f is Darboux integrable if $U(f) = L(f)$. Here we take random sums $R(f, P)$ and say that the sums converge to a common value.

This makes the following result even more surprising:

Theorem:

f is Darboux integrable $\Leftrightarrow$ f is Riemann integrable

In that case, the Darboux integral and the Riemann integral are the same.

Recall:

$$M(f,[t_{k-1},t_k]) = \sup\{\, f(x) \mid x \text{ in } [t_{k-1},t_k]\,\}$$
$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1})$$
$$U(f) = \inf\{\, U(f,P) \mid P \text{ is a partition of } [a,b]\,\}$$

Idea of Proof: (The details are in the book)

($\Leftarrow$) For any partition P, since $L(f,P)$ is an underestimate and $U(f,P)$ is an overestimate, we have $L(f,P) \le R(f,P) \le U(f,P)$ and since $L(f) = U(f) = \int_a^b f(x)dx$, intuitively forces all the Riemann sums $R(f,P)$ to have the same limit $\int_a^b f(x)dx$.

($\Rightarrow$) Here you just *choose* x_k to give you the upper sum $U(f,P)$, and similarly you *choose* x_k to give you the lower sum $L(f,P)$, so since by assumption every Riemann sum $R(f,P)$is the same, this forces $L(f) = U(f)$

2. Cauchy Criterion

In practice, it is a pain to calculate $U(f)$ and $L(f)$; it's *much* easier to deal with $U(f,P)$ and $L(f,P)$, since we can concretely calculate them. Luckily there's a way to talk about integrability without explicitly mentioning $U(f)$.

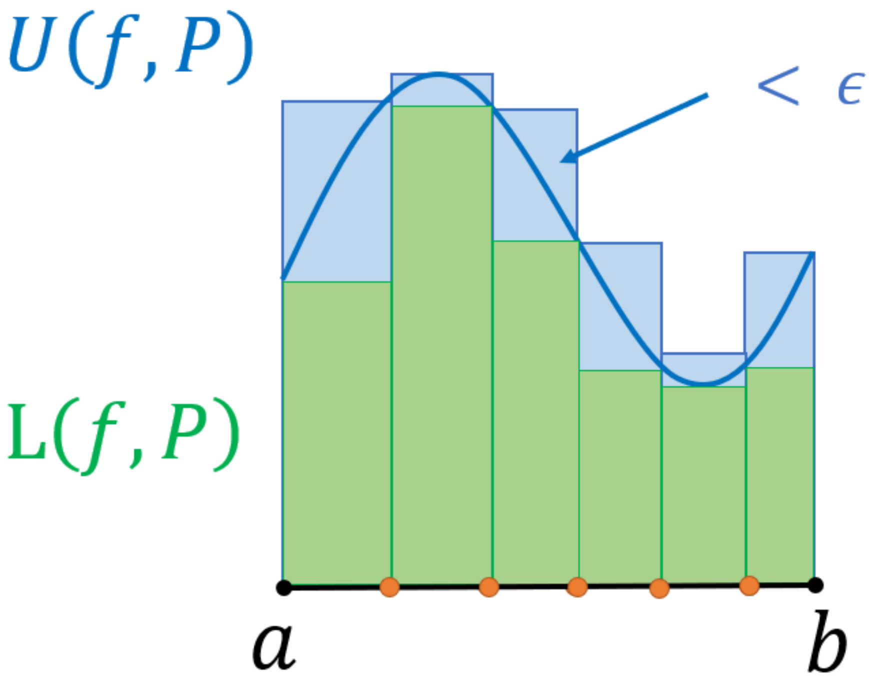
The following should remind you of the Cauchy criterion for series:

Cauchy Criterion for integrals:

f is integrable if and only if for all $\epsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon$$

"No matter how small, we can always find a partition that makes the difference (in blue) as small as we want"



Proof:

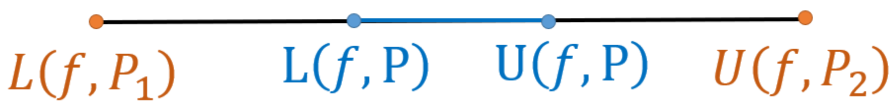
($\Rightarrow$) Let $\epsilon > 0$ be given, then and consider:

$$L(f) - \frac{\epsilon}{2} < L(f) = \sup \{ \, L(f, P) \mid P \text{ partition } \}$$

By def of sup, there is a partition P_1 such that $L(f, P_1) > L(f) - \frac{\epsilon}{2}$

Similarly there is a partition P_2 such that $U(f, P_2) < U(f) + \frac{\epsilon}{2}$

Let $P = P_1 \cup P_2$ (finer), then $L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2)$, and therefore:



$$\begin{aligned} U(f,P) - L(f,P) \leq & U(f,P_2) - L(f,P_1) \\ < & U(f) + \frac{\epsilon}{2} - \left(L(f) - \frac{\epsilon}{2}\right) \\ = & \underbrace{U(f) - L(f)}_{0} + \epsilon \\ = & \epsilon \end{aligned}$$

Here we used $U(f) = L(f)$, since f is integrable ✓

($\Leftarrow$) Let $\epsilon > 0$ be given and let P be such that $U(f, P) - L(f, P) < \epsilon$. Then by definition of $U(f)$ as an inf, we get:

$$\begin{aligned} U(f) \leq & U(f,P) \\ = & U(f,P) - L(f,P) + L(f,P) \\ < & \epsilon + L(f,P) \\ \leq & \epsilon + L(f) \end{aligned}$$

Hence $U(f) < L(f) + \epsilon$ for all $\epsilon > 0$, hence $U(f) \leq L(f)$, but since $L(f) \leq U(f)$ as well, we get $U(f) = L(f)$ ✓ $\square$

3. Integrability and Monotonicity

Here are two nice applications of the Cauchy criterion. First, let's show that monotonic functions are integrable. Recall that monotonic means either increasing or decreasing.

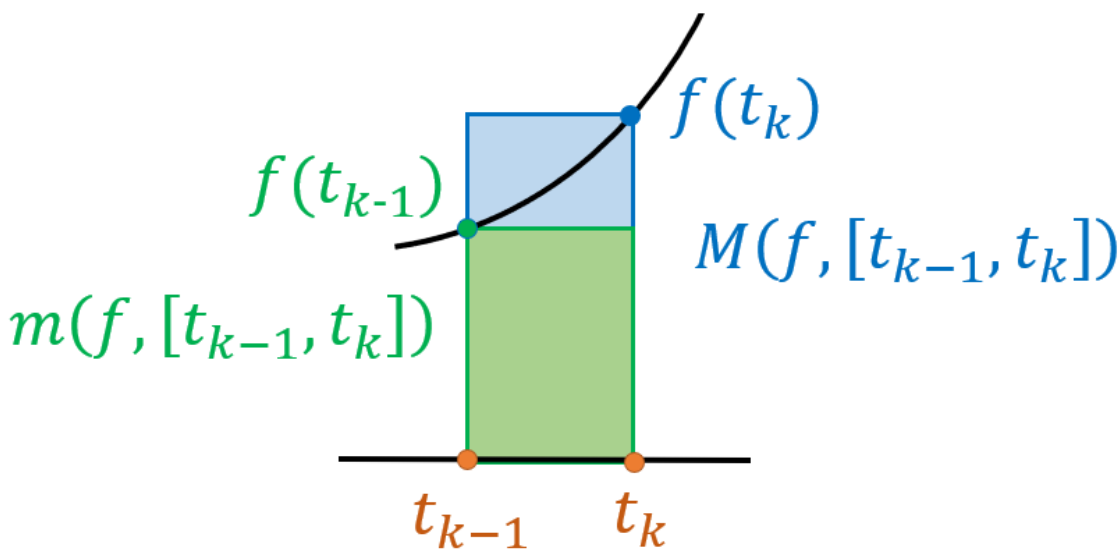
Theorem:

If f is monotonic on $[a, b]$, then f is integrable on $[a, b]$

Proof: WLOG, assume f is strictly increasing, and so $f(a) < f(b)$

Main Observation: (just like last time with $f(x) = x^2$). In that case, we have

$$M(f, [t_{k-1}, t_k]) = f(t_{k-1}) \text{ and } m(f, [t_{k-1}, t_k]) = f(t_k)$$



In order to show f is integrable, let's use the Cauchy criterion above.

Let $\epsilon > 0$ be given, let $\delta = \frac{\epsilon}{f(b)-f(a)}$ and and let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be any partition with mesh $< \delta$, then:

$$\begin{aligned}
U(f,P) - L(f,P) =& \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) - \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1}) \\
=& \sum_{k=1}^{n} f(t_k)(t_k - t_{k-1}) - \sum_{k=1}^{n} f(t_{k-1})(t_k - t_{k-1}) \\
=& \sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1})\right)(t_k - t_{k-1}) \\
<& \sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1})\right) \frac{\epsilon}{f(b)-f(a)} \\
=& \frac{\epsilon}{f(b)-f(a)} \sum_{k=1}^{n} f(t_k) - f(t_{k-1}) \\
=& \left(\frac{\epsilon}{f(b)-f(a)}\right)(f(t_n) - f(t_0)) \qquad \text{(Telescoping sum)} \\
=& \left(\frac{\epsilon}{f(b)-f(a)}\right)(f(b) - f(a)) \\
=& \epsilon \checkmark
\end{aligned}$$

Hence f is integrable $\square$

4. Continuity and Integrability

Here is a second application of the Cauchy criterion:

Theorem:

If f is continuous on $[a,b]$, then f is integrable on $[a,b]$

This is why in calculus you mainly integrate continuous functions.

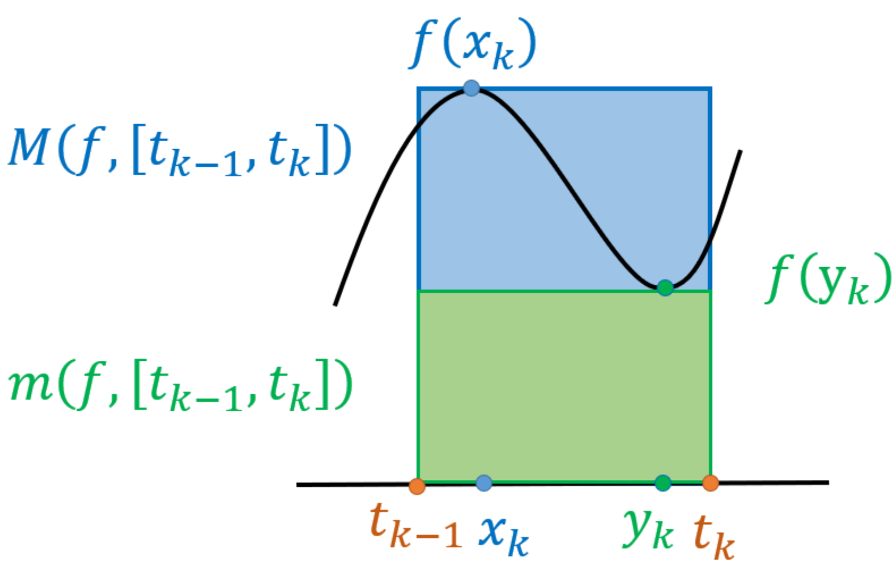
Proof: Beautiful application of uniform continuity!

Since f is continuous on $[a,b]$, it is uniformly continuous on $[a,b]$

Let $\epsilon > 0$ be given, then there is $\delta > 0$ such that for all x and y, if $|x-y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{b-a}$

Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be any part. with $\text{mesh}(P) < \delta$.

Since f is continuous on each sub-piece $[t_{k-1}, t_k]$, it attains a maximum and a minimum for some x_k and y_k in $[t_{k-1}, t_k]$



Therefore, by definition,

$$M(f,[t_{k-1},t_k]) = f(x_k) \text{ and } m(f,[t_{k-1},t_k]) = f(y_k)$$

But then we get:

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{k=1}^{n} \left(M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\right)(t_k - t_{k-1}) \\
&= \sum_{k=1}^{n} \left(f(x_k) - f(y_k)\right)(t_k - t_{k-1}) \\
&\leq \sum_{k=1}^{n} |f(x_k) - f(y_k)|\,(t_k - t_{k-1}) \\
&< \sum_{k=1}^{n} \left(\frac{\epsilon}{b-a}\right)(t_k - t_{k-1}) \qquad \text{(Uniform Continuity)} \\
&= \frac{\epsilon}{b-a} \sum_{k=1}^{n} t_k - t_{k-1} \\
&= \left(\frac{\epsilon}{b-a}\right)(b-a) \\
&= \epsilon \checkmark
\end{aligned}$$

Hence, by the Cauchy Criterion, f is integrable on $[a,b]$ □

Notice the crucial role that uniform continuity played here, since we don't know where the x_k and y_k are (relative to $[a,b]$)

Note: The same two results hold if f is only piecewise monotonic or piecewise continuous, although with different proofs.

So far we've seen sufficient conditions for integrability, which might lead you to ask: Is there a more general theorem that tells us **exactly** when a function is integrable? The answer is YES!!

Riemann-Lebesgue Theorem:

A bounded function f is integrable on $[a, b]$ if and only if the set of discontinuities of f has "measure 0"

Note: Do **NOT** use this theorem on the homework or exams, unless you also provide me a proof ☺

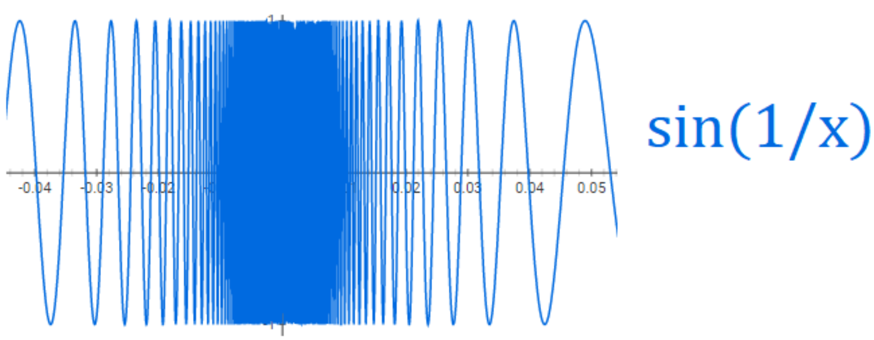
Here measure 0 means "negligible," think "probability 0" So a finite set or even $\mathbb{Q}$ has measure 0: If you pick a real number at random, the probability that it is rational is 0.

Example 1:

Continuous functions are integrable (no discontinuities)

Example 2:

$f(x) = \sin\left(\frac{1}{x}\right)$ is integrable on $[-1, 1]$ (only one discontinuity)



Example 3:

The Popcorn function (from the section 17 homework) is integrable since it is discontinuous only at the rational numbers