LECTURE 26: TRIPLE INTEGRALS (II)

1. Volumes

Video: Volume of Gouda Cheese

Even though *in general* a triple integral doesn't calculate a volume, there is one special case where it does:

Fact:

$$\mathrm{Vol}(E) = \int\int\int_E 1 \, dxdydz$$

Note: Use this to calculate volumes, instead of $\int\int_D$ Big − Small

Example 1:

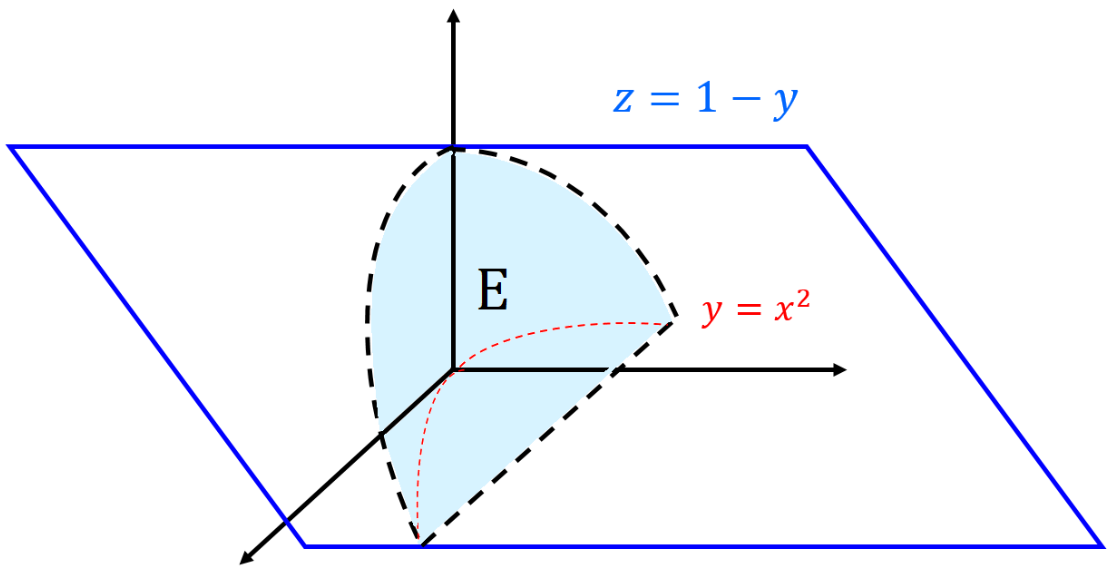
Find Vol(E) where E is the region enclosed by the surfaces

$$\begin{cases} y = x^2 \\ z = 1 - y \\ z = 0 \end{cases}$$

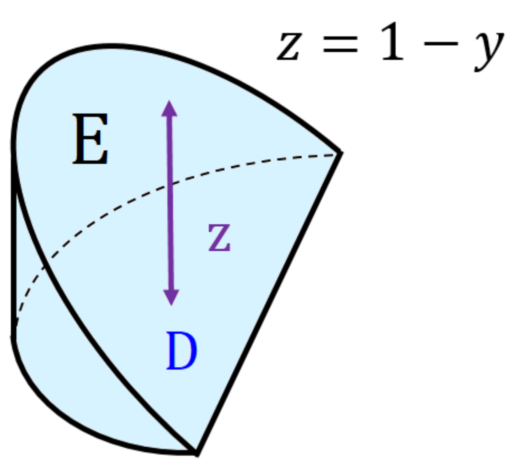
STEP 1: Picture:

Date: Wednesday, October 27, 2021.

Suggestion: First start with $z = 1 - y$, which is a plane in the x direction (since x is missing). Then cut the plane along the parabola $y = x^2$.



Re-Draw:

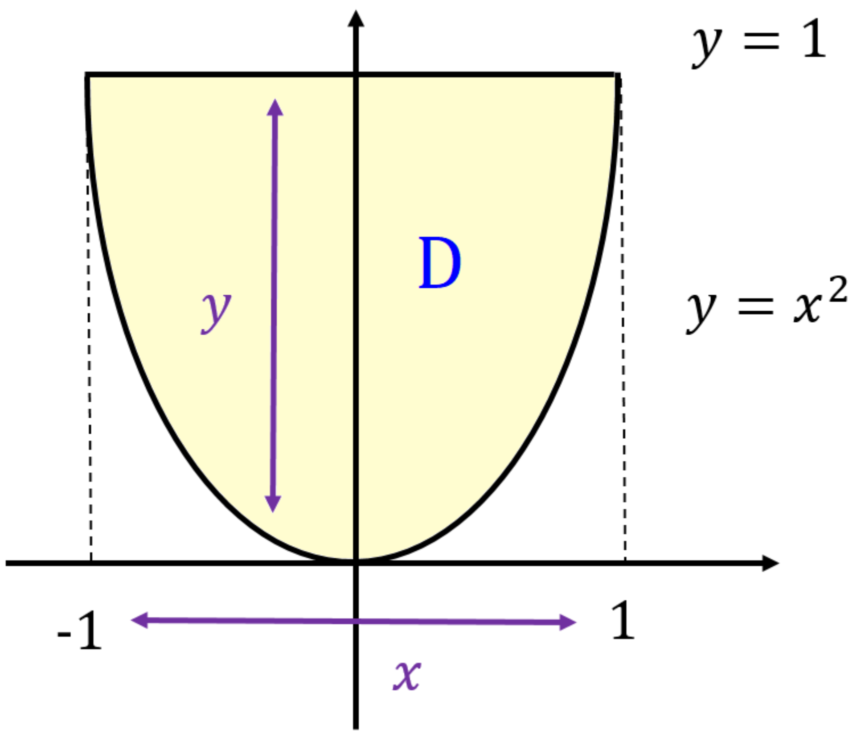


STEP 2: Inequalities:

$$\text{Small } \leq z \leq \text{ Big}$$
$$0 \leq z \leq 1 - y$$

STEP 3: Find D

Note: Notice $z = 0$ in D, so $z = 1 - y \Rightarrow 0 = 1 - y \Rightarrow y = 1$ (which is a straight line)



$$\text{Small } \leq y \leq \text{ Big}$$
$$x^2 \leq y \leq 1$$

Finally notice that $x^2 = 1 \Rightarrow x = \pm 1$, so

$$-1 \leq x \leq 1$$

Therefore our inequalities become

$$\begin{cases} 0 \leq z \leq 1 - y \\ x^2 \leq y \leq 1 \\ -1 \leq x \leq 1 \end{cases}$$

STEP 4: Integrate

$$\begin{aligned}
\text{Vol}(E) &= \int\int\int_E 1\, dxdydz \\
&= \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} 1\, dzdydx \\
&= \int_{-1}^{1}\int_{x^2}^{1} 1 - y\, dydx \\
&= \int_{-1}^{1} \left[y - \frac{y^2}{2}\right]_{y=x^2}^{y=1} \\
&= \int_{-1}^{1} 1 - \frac{1}{2} - x^2 + \frac{x^4}{2} dx \\
&= \int_{-1}^{1} \frac{x^4}{2} - x^2 + \frac{1}{2} dx \\
&= 2\int_{0}^{1} \frac{x^4}{2} - x^2 + \frac{1}{2} dx \qquad \text{(The function is even)} \\
&= 2\left[\frac{x^5}{10} - \frac{x^3}{3} + \frac{x}{2}\right]_0^1 \\
&= \frac{8}{15}
\end{aligned}$$

Warning: For volume questions should never get 0 or a negative answer!

2. Other Directions

Video: Integral over Cannoli

From the creator of the band *One Direction* comes the spin-off called *Other Directions*

Just like double integrals where you can do horizontal regions, here you can also do triple integrals in different directions.

Example 2:

Calculate the following integral

$$\int\int\int_E 3\,dxdydz$$

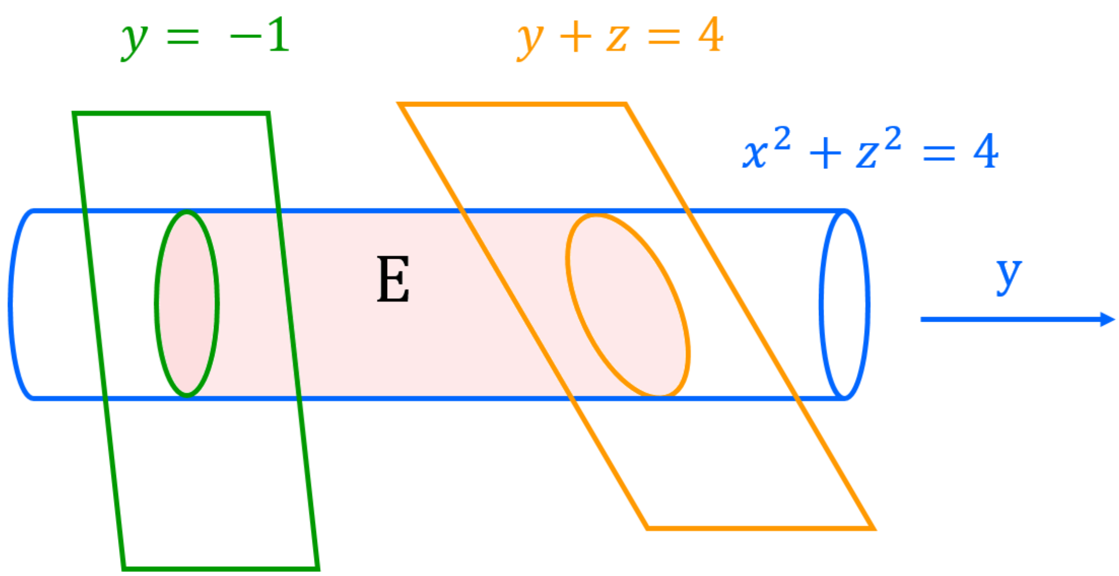
Where E is the solid enclosed by the following surfaces:

$$\begin{cases} x^2 + z^2 = 4 \\ y = -1 \\ y + z = 4 \end{cases}$$

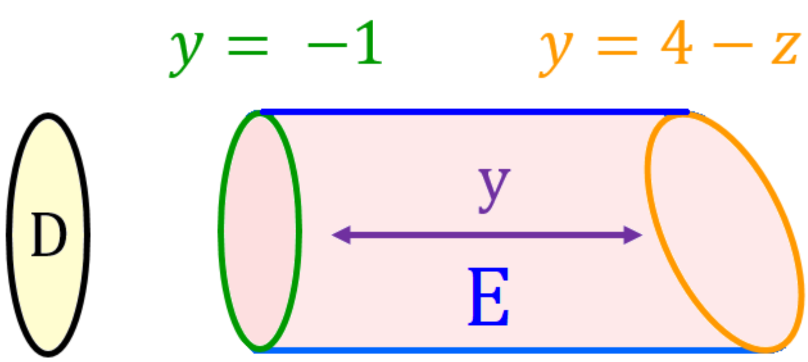
STEP 1: Picture:

$x^2 + z^2 = 4$ is a cylinder, but in the **y**-direction (because y is missing)

$y+z=4$ is a plane, but in the x-direction (to draw this, draw the line $y+z=4$ and move it along the x axis)



Re-Draw:



Here the region is in the $y-$direction (Book calls this a Type 2 region)

STEP 2: Inequalities:

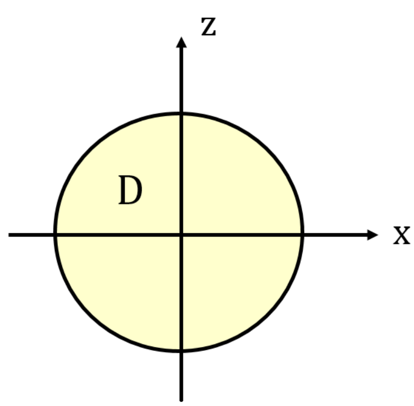
Usually you do Small $\leq z \leq$ Big, but since everything is in the $y-$direction, this time it's:

$$\begin{aligned} \text{Left } &\leq y \leq \text{ Right} \\ -1 &\leq y \leq 4 - z \end{aligned}$$

Note: This makes sense if you tilt your head in the $y-$direction and see which function is above and below you!

STEP 3: Find D

Here D is the shadow to the left of E, which here is a disk of radius 2 in x and z



$$\begin{aligned} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{aligned}$$

Notice in particular here that $x = r\cos(\theta)$ and $z = r\sin(\theta)$

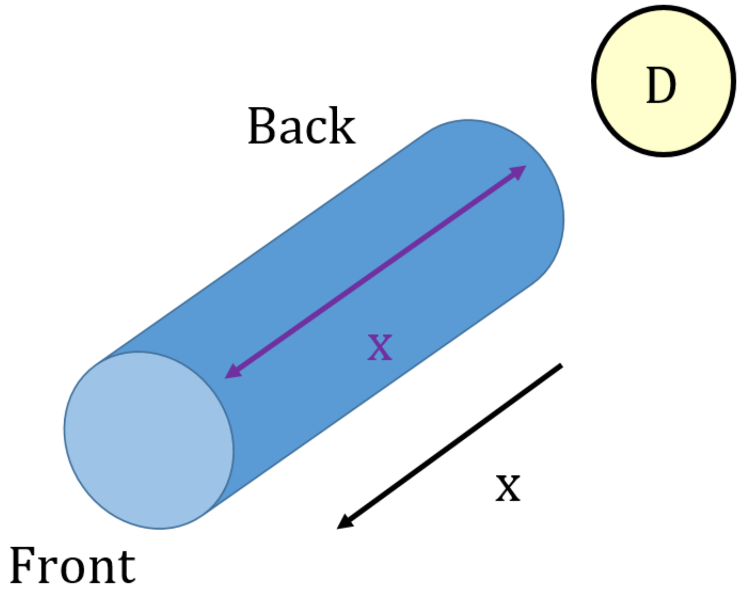
Therefore our inequalities are

$$\begin{cases} -1 \leq y \leq 4 - z \\ 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

STEP 4: Integrate

$$\begin{aligned}
\int\int\int_E 3\, dxdydz &= \int\int_D \int_{-1}^{4-z} 3\, dydxdz \\
&= \int\int_D 3\left((4-z) - (-1)\right) dxdz \\
&= \int\int_D 3\,(5-z)\, dxdz \\
&= \int\int_D 15 - 3z\, dxdz \\
&= \int_0^{2\pi}\int_0^2 (15 - 3r\sin(\theta))\, rdrd\theta \\
&= \int_0^{2\pi}\int_0^2 15r - 3r^2\sin(\theta) drd\theta \\
&= \int_0^{2\pi} \left[\frac{15}{2}r^2 - r^3\sin(\theta)\right]_{r=0}^{r=2} d\theta \\
&= \int_0^{2\pi} 30 - 8\sin(\theta) d\theta \\
&= \left[30\theta + 8\cos(\theta)\right]_0^{2\pi} \\
&= 30(2\pi) \\
&= 60\pi
\end{aligned}$$

Note: Sometimes your surface faces the $x-$direction, as in the following picture



In that case, we have Back $\leq x \leq$ Front and D is the shadow behind the surface.

3. Averages

Similar to before, we can find the average value of a 3D function:

Note: For double integrals, we had to divide by Area (D), and now for triple integrals, we have to divide by the volume:

Definition:

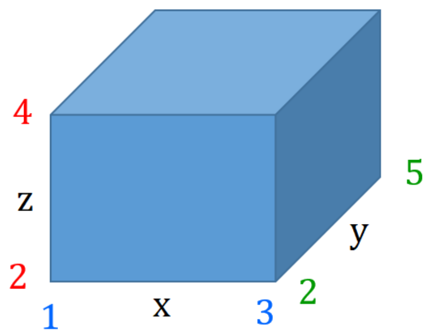
The average value of $f(x, y, z)$ over a solid E is

$$\frac{1}{\text{Vol}(E)} \int\int\int_E f(x, y, z) dx dy dz$$

Example 3:

Find the average value of $f(x, y, z) = xyz$ over the box

$$E = [1, 3] \times [2, 5] \times [2, 4]$$



$$\begin{aligned}
&\int\int\int_E xyzdxdydz \\
=&\int_2^4\int_2^5\int_1^3 xyz\,dxdydz \\
=&\left(\int_1^3 xdx\right)\left(\int_2^5 ydy\right)\left(\int_2^4 zdz\right) \\
=&\left[\frac{x^2}{2}\right]_1^3\left[\frac{y^2}{2}\right]_2^5\left[\frac{z^2}{2}\right]_2^4 \\
=&\left(\frac{9-1}{2}\right)\left(\frac{25-4}{2}\right)\left(\frac{16-4}{2}\right) \\
=&\left(\frac{8}{2}\right)\left(\frac{21}{2}\right)\left(\frac{12}{2}\right) \\
=&21 \times 12 \\
=&252
\end{aligned}$$

$$\text{Vol }(E) = (3-1)\times(5-2)\times(4-2) = 2\times 3\times 2 = 12$$

$$\text{Average } = \frac{252}{12} = \frac{21\times 12}{12} = 21$$

4. Intersection of two cylinders

Video: Volume of Intersection of two cylinders

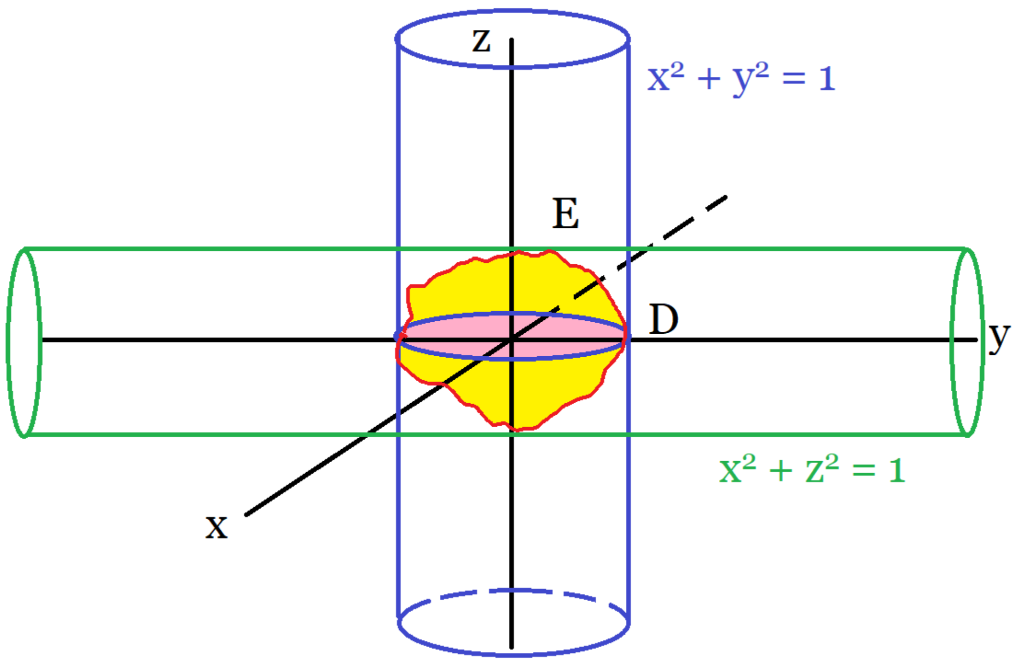
Here is a fun challenge problem that math can sometimes solve things our eyes cannot see!

Example 4:

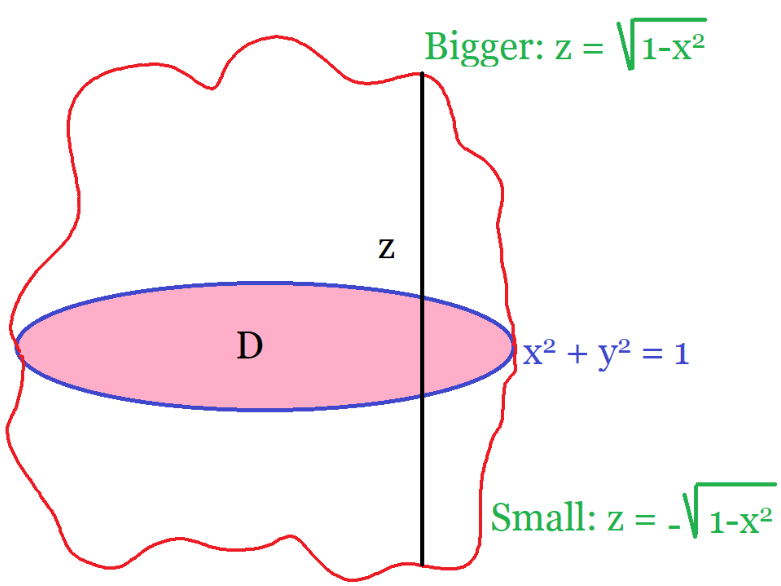
Find the volume of the intersection of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$

STEP 1: Picture:

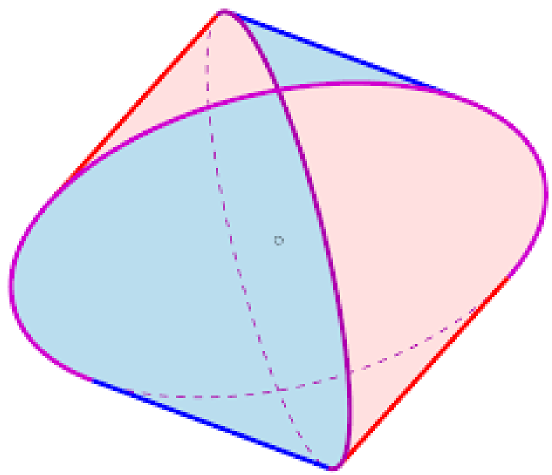
$x^2 + y^2 = 1$ is a cylinder in the $z-$direction, and $x^2 + z^2 = 1$ is a cylinder in the $y-$direction.



Problem: E is really hard to visualize! In that case: Believe in the math, not your eyes!



Note: If you're curious what it actually looks like, here's a picture.



STEP 2: Inequalities:

$$\text{Small } \leq z \leq \text{ Big}$$

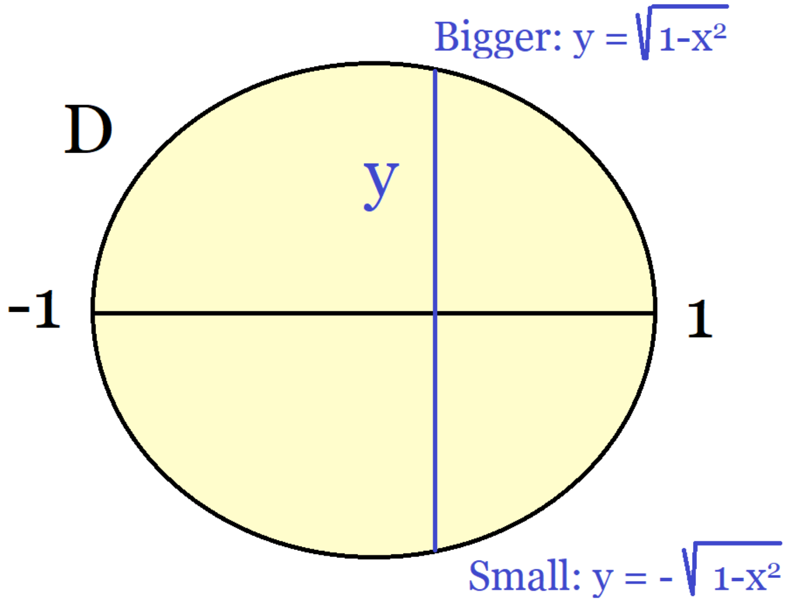
$$z^2 + x^2 = 1 \Rightarrow z^2 = 1 - x^2 \Rightarrow z = \pm\sqrt{1 - x^2}$$

$$-\sqrt{1 - x^2} \leq z \leq \sqrt{1 - x^2}$$

Note: Why use $z^2 + x^2 = 1$? It's the only equation with z! Also it makes sense in terms of the first picture and it's the direction that makes D the easiest.

STEP 3: Find D

Based on the pictures above, D is a disk of radius 1 (you can get that by setting $z = 0$ in $x^2 + y^2 = 1$)



STEP 4: Warning: You *could* use polar coordinates here, but if you do that (and I invite you to try it out), it becomes a **HUGE** mess, so instead go back to the bigger and smaller technique:[1]

$$\text{Small } \leq y \leq \text{ Big}$$

$$x^2 + y^2 = 1 \Rightarrow y^2 = 1 - x^2 \Rightarrow -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$$

$$\begin{cases} -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \\ -1 \leq x \leq 1 \end{cases}$$

[1] On the exam, I would give you a hint not to use polar coordinates

STEP 5: Integrate:

$$
\begin{aligned}
\mathrm{Vol}(E) &= \int\int\int_E 1\, dxdydz \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dzdydx \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) dydx \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2}\, dydx \\
&= \int_{-1}^{1} 2\sqrt{1-x^2}\left(\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right) dx \\
&= \int_{-1}^{1} \left(2\sqrt{1-x^2}\right)\left(2\sqrt{1-x^2}dx\right) \\
&= \int_{-1}^{1} 4(1-x^2)dx \\
&= 2\int_{0}^{1} 4\left(1-x^2\right) dx \qquad \text{(The function is even)} \\
&= \frac{16}{3}
\end{aligned}
$$

Note: If you're curious how to find the volume of the intersection of 3 cylinders, check out the following optional video:

Optional Video: Volume of Intersection of three cylinders

5. Changing the order of integration

Note: I will **NOT** ask about this on the quiz or exams, but here is how to change the order of integration in a *triple* integral.

Example 5:

Write the integral in the following order

$$\int_0^{16}\int_{\sqrt{x}}^{4}\int_0^{4-y} f(x,y,z)dzdydx = \int_?^?\int_?^?\int_?^? f(x,y,z)dydxdz$$

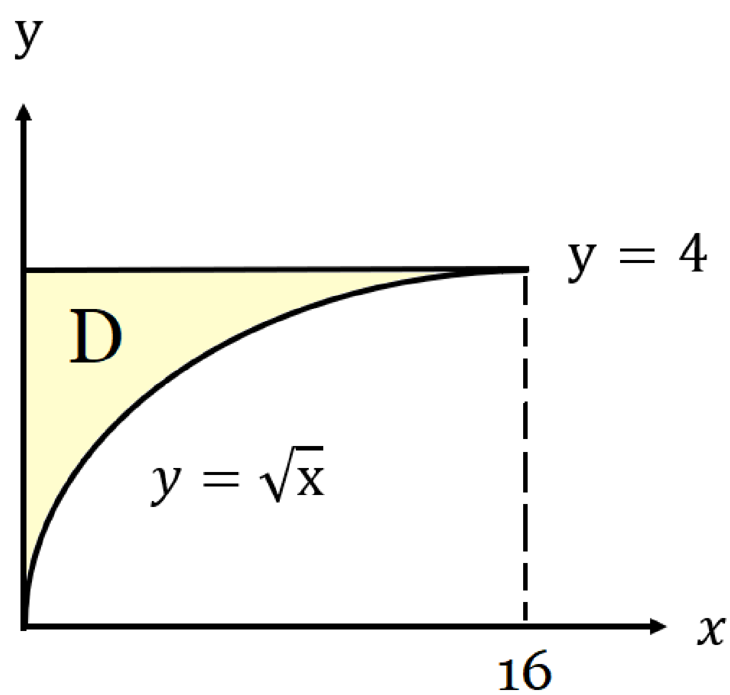
STEP 1: Inequalities

$$\begin{cases} 0 \leq z \leq 4-y \\ \sqrt{x} \leq y \leq 4 \\ 0 \leq x \leq 16 \end{cases}$$

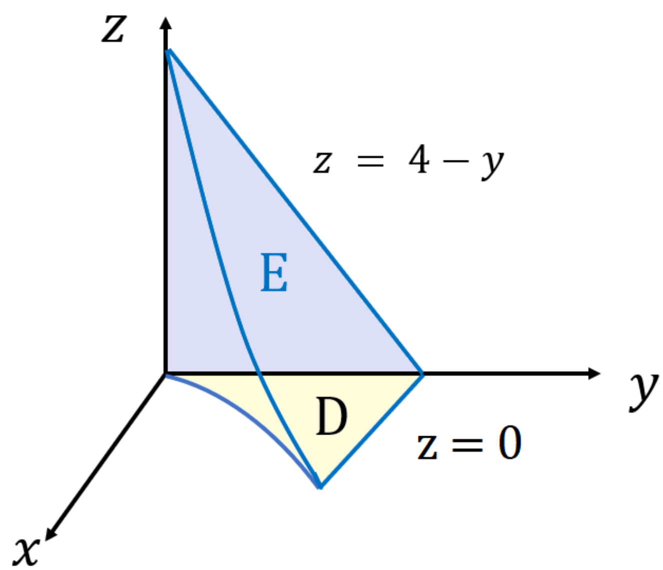
This says that the bigger function is $z = 4-y$ and the smaller is $z = 0$

STEP 2: Draw D:

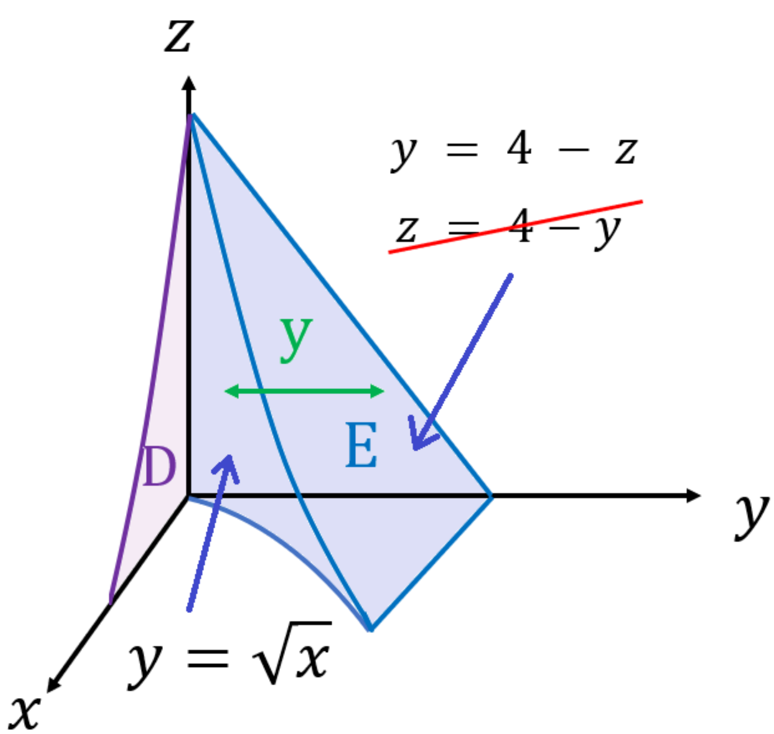
$$\begin{cases} \sqrt{x} \leq y \leq 4 \\ 0 \leq x \leq 16 \end{cases}$$



STEP 3: **Draw** E**:**



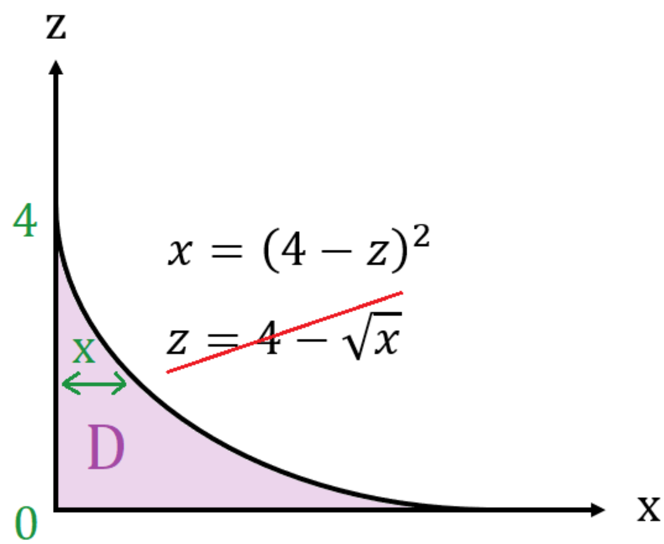
STEP 4: We want the integral in the form $dydxdz$, so first we want y in terms of x and z, then x in terms of z, and then z constant



Based on the picture above, the left function is $y = \sqrt{x}$ and the right function is $z = 4 - y \Rightarrow y = 4 - z$

$$\text{Left } \leq y \leq \text{ Right}$$
$$\sqrt{x} \leq y \leq 4 - z$$

To find D, which is the shadow to the left of the region, notice that if $z = 4 - y$ and $y = \sqrt{x}$, we get $z = 4 - \sqrt{x}$, which is precisely the curved part of D:



Finally, since we want y in terms of x and z, we need to write this as a horizontal region, and since $z = 4 - \sqrt{x} \Rightarrow x = (4 - z)^2$, we get

$$\begin{aligned} 0 \leq & x \leq (4-z)^2 \\ 0 \leq & z \leq 4 \end{aligned}$$

STEP 5: Answer

$$\begin{cases} \sqrt{x} \leq y \leq 4 - z \\ 0 \leq x \leq (4-z)^2 \\ 0 \leq z \leq 4 \end{cases}$$

$$\int_0^{16} \int_{\sqrt{x}}^{4} \int_0^{4-y} f(x,y,z)\,dzdydx = \int_0^4 \int_0^{(4-z)^2} \int_{\sqrt{x}}^{4-z} f(x,y,z)\,dydxdz$$

Good luck doing this in all 6 different orders ☺

It's not as bad as you think, since you already did the hardest part, which is drawing E.