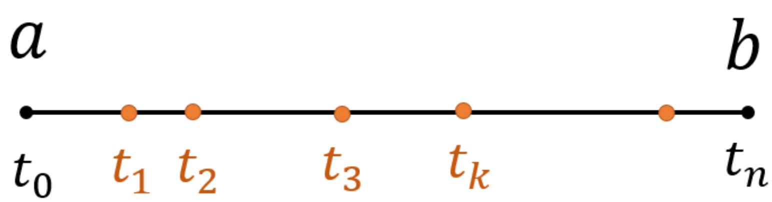
# LECTURE 26: PROPERTIES OF THE INTEGRAL (II)

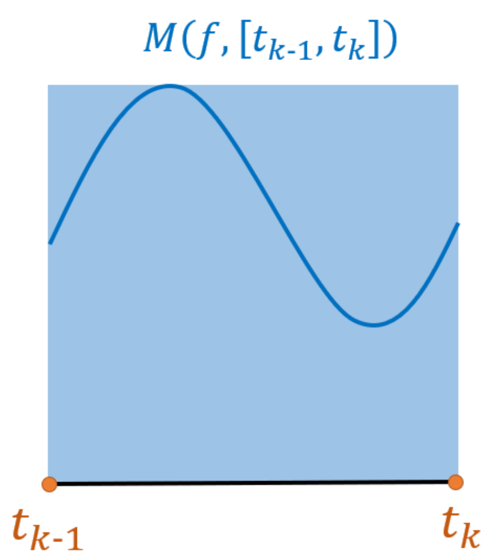
## 1. Recap

(1) **Partition**

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$




(2) **Max on sub-piece**




*Date*: Tuesday, November 30, 2021.

$$M(f, [t_{k-1}, t_k]) = \sup \{ \, f(x) \mid x \text{ in } [t_{k-1}, t_k] \, \}$$

(3) **Upper Sum**

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

(4) **Upper Integral**

Adding rectangles causes $U(f, P)$ to *decrease*, and so:

$$U(f) = \inf \{ \, U(f, P) \mid P \text{ is a partition of } [a, b] \, \}$$

Similarly we have lower sum $L(f, P)$ and lower integral $L(f)$

(5) **Darboux Integral**

$f$ is **integrable** on $[a, b]$ if $L(f) = U(f)$

Finally, there is the Cauchy criterion, which is useful if you don't know what the integral is:

**Cauchy Criterion for integrals:**

$f$ is integrable if and only if for all $\epsilon > 0$ there is a partition $P$ of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon$$

## 2. $f + g$ AND $cf$

**Theorem:**

If $f$ is integrable on $[a, b]$ then so is $cf$, and $\int_a^b cf = c \int_a^b f$

**Sketch of Proof:**

If $c > 0$ this follows from $M(cf, [t_{k-1}, t_k]) = cM(f, [t_{k-1}, t_k])$, and taking sums we get $U(cf, P) = cU(f, P)$, and taking inf we get $U(cf) = cU(f)$.

For $c = -1$ you use $U(-f, P) = -L(f, P)$ and then take inf, compare this with with $\inf(S) = -\sup(-S)$

Finally for $c < 0$ you use $c = -\underbrace{(-c)}_{>0}$ and the above two steps □

**Theorem:**

If $f$ and $g$ are integrable on $[a, b]$ then so is $f + g$, and

$$\int_a^b f + g = \int_a^b f + \int_a^b g$$

**Proof:**

**STEP 1:** The main idea is to use

$$\sup \{f(x) + g(x)\} \leq \sup \{f(x)\} + \sup \{g(x)\}$$

By definition of $M$ as a sup, this implies that

$$M(f + g, [t_{k-1}, t_k]) \leq M(f, [t_{k-1}, t_k]) + M(g, [t_{k-1}, t_k])$$

And therefore taking sums, we get $U(f + g, P) \leq U(f, P) + U(g, P)$

Similarly $L(f + g, P) \geq L(f, P) + L(g, P)$

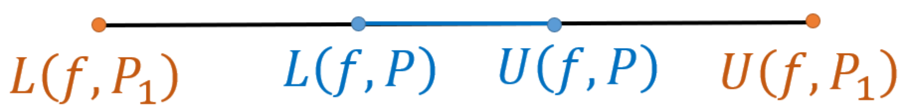
**STEP 2:** The idea now is to use the Cauchy criterion:

Let $\epsilon > 0$ be given, then since $f$ and $g$ are integrable, there are partitions $P_1$ and $P_2$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2} \text{ and } U(g, P_2) - L(g, P_2) < \frac{\epsilon}{2}$$

We would like to use a common partition, so let $P = P_1 \cup P_2$


Since $P$ is finer than both $P_1$ and $P_2$, we have $U(f, P) \leq U(f, P_1)$ and $L(f, P) \geq L(f, P_1)$




$$\text{Therefore: } U(f, P) - L(f, P) \leq U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2}$$

Similarly, we have $U(g, P) - L(g, P) < \frac{\epsilon}{2}$

**STEP 3:** From $U(f+g, P) \leq U(f, P) + U(g, P)$ and $L(f+g, P) \geq L(f, P) + L(g, P)$ we get:

$$\begin{aligned} U(f+g, P) - L(f+g, P) &\leq (U(f, P) + U(g, P)) - (L(f, P) + L(g, P)) \\ &= U(f, P) - L(f, P) + U(g, P) - L(g, P) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad \text{(From } \textbf{STEP 2}\text{)} \\ &= \epsilon \end{aligned}$$

Hence by the Cauchy Criterion, $f+g$ is integrable on $[a,b]$

**STEP 4:** To evaluate the integral, we use:

$$\begin{aligned}\int_a^b f+g =& U(f+g) \\ \leq& U(f+g,P) && \text{Since } U(f+g) \text{ is the inf of } U(f+g,P) \\ \leq& U(f,P)+U(g,P) && \text{(By \textbf{STEP 1})} \\ <& L(f,P)+\frac{\epsilon}{2}+L(g,P)+\frac{\epsilon}{2} && \text{(By \textbf{STEP 2})} \\ \leq& L(f)+L(g)+\epsilon && \text{Since } L(f) \text{ is the sup of } L(f,P) \\ =& \left(\int_a^b f+\int_a^b g\right)+\epsilon \end{aligned}$$

So $\int_a^b f+g \leq \left(\int_a^b f+\int_a^b g\right)+\epsilon$ and since $\epsilon>0$ was arbitrary, we get $\int_a^b f+g \leq \int_a^b f+\int_a^b g$

Similarly, using $L(f+g)$, we get $\int_a^b f+g \geq \int_a^b f+\int_a^b g$

And therefore $\int_a^b f+g = \int_a^b f+\int_a^b g$ ✓ □

## 3. More Properties

Here are some properties of integrals frequently used in calculus:

**Fact 1:**

$$f(x) \geq 0 \text{ for all } x \;\Rightarrow \int_a^b f \geq 0$$

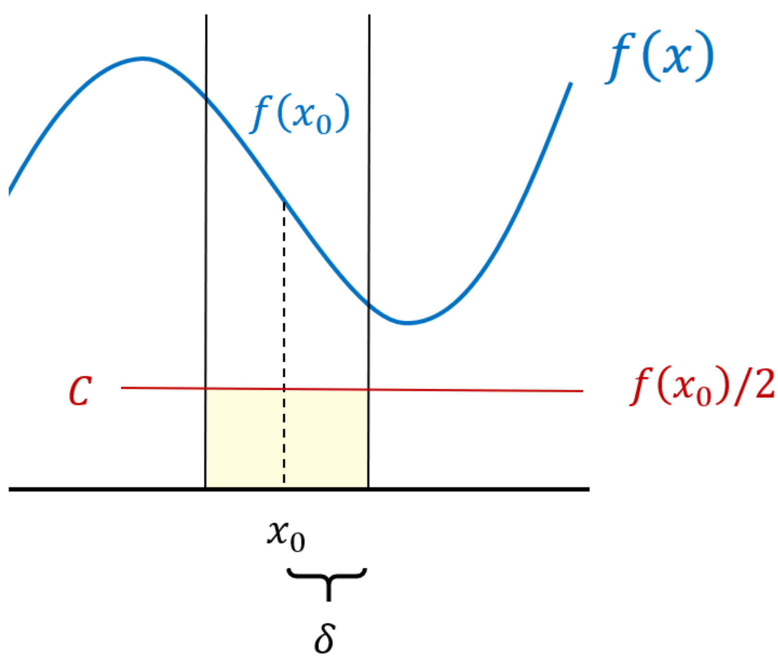
**Proof:** By assumption $f(x) \geq 0$ for all $x$ in $[a, b]$ and so for all partitions $P$, we have $L(f, P) \geq 0$. Taking the sup over all partitions $P$, we get $L(f) \geq 0$ and so since $f$ is integrable (by assumption), we get $\int_a^b f = L(f) \geq 0$ □

From this it follows that if $f \leq g$ then $\int_a^b f \leq \int_a^b g$ (simply by considering $h = g - f$)

**Fact 2:**

If $f \geq 0$ is continuous and $\int_a^b f = 0$, then $f(x) = 0$ for all $x$

**Proof:** Suppose $f(x_0) \neq 0$ for some $x_0$, then WLOG, $f(x_0) > 0$




Since $f$ is continuous at $x_0$ there is some $\delta > 0$ such that $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \frac{f(x_0)}{2}$, which implies that:

$$-\frac{f(x_0)}{2} < f(x) - f(x_0) < \frac{f(x_0)}{2} \Rightarrow f(x) > f(x_0) - \frac{f(x_0)}{2} = \frac{f(x_0)}{2}$$

In particular, on the interval $(x_0 - \delta, x_0 + \delta)$, we have $f(x) > C$, where $C = \frac{f(x_0)}{2} > 0$ and so

$$\int_a^b f \geq \int_{x_0-\delta}^{x_0+\delta} f > \int_{x_0-\delta}^{x_0+\delta} C = C(x_0 + \delta - (x_0 - \delta)) = C(2\delta) > 0$$

Which contradicts $\int_a^b f = 0 \Rightarrow\Leftarrow$ □

A similar argument shows that if $\int_a^b fg = 0$ for all $g$, then $f = 0$ everywhere. This is useful in more advanced analysis.

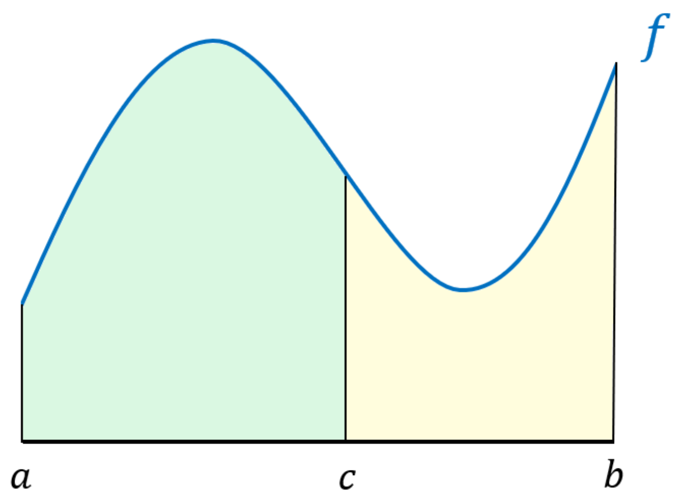
**Fact 3: (Triangle Inequality)**

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

**Proof-Sketch:** This simply follows from $-|f| \leq f \leq |f|$ and integrating. Of course we would also need to show that $|f|$ is integrable, but this is an application of the Cauchy criterion, see book

**Fact 4:**

$$\int_a^b f = \int_a^c f + \int_c^b f$$




**Proof-Sketch:** See book, but basically you consider two partitions, one on $[a, c]$ and another on $[c, b]$, take the union, and use the Cauchy criterion, similar to what we did with $f + g$

## 4. Average Value

What does it mean to calculate the average grade in a class? You take the sum of grades and then divide by the number of students. For integrals it is the same thing:

**Definition**

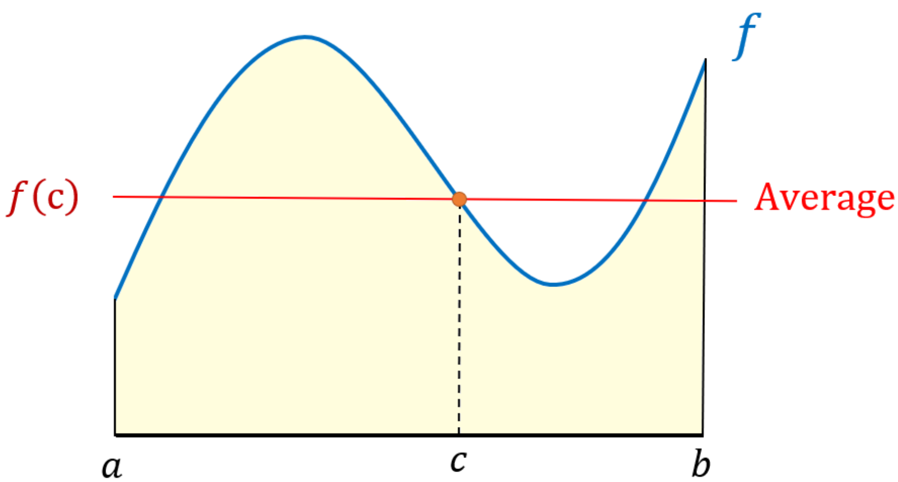
The average value of $f$ on $[a, b]$ is $\frac{\int_a^b f}{b-a}$

Although in the discrete setting, the average value might not be attained, for functions, it always is:

## MVT for Integrals

If $f$ is continuous, then there is at least one $c$ in $(a, b)$ such that

$$\frac{\int_a^b f}{b-a} = f(c)$$




So, in the world of functions, if the average grade is 50, there is a student who actually got 50. Or if your average speed was 65 mph, then you actually drove 65 mph at some point

**Proof:** Here we'll cheat a bit and use the Fundamental Theorem of Calculus, which we'll cover next time

$$\text{Let } F(x) = \int_a^x f(t)dt$$

(Which is defined since $f$ is continuous)

Then by the regular MVT applied to $F$, for some $c$ in $(a, b)$, we have

$$\frac{F(b) - F(a)}{b - a} = F'(c)$$

But $F(b) = \int_a^b f(t)dt$, $F(a) = \int_a^a f(t)dt = 0$ and by the FTC, $F'(x) = f(x)$, and so $F'(c) = f(c)$

Therefore the above becomes

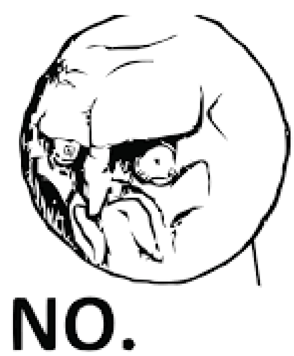
$$\frac{\int_a^b f(t)dt}{b - a} = f(c)$$

Which is what we wanted to show $\square$

## 5. Integrals and Limits

Finally, let's answer a question that has haunted all of math-kind for centuries: Is it ok to put the limit inside the integral? In other words, it is true that

$$\lim_{n \to \infty} \int_a^b f_n \stackrel{?}{=} \int_a^b \lim_{n \to \infty} f_n$$

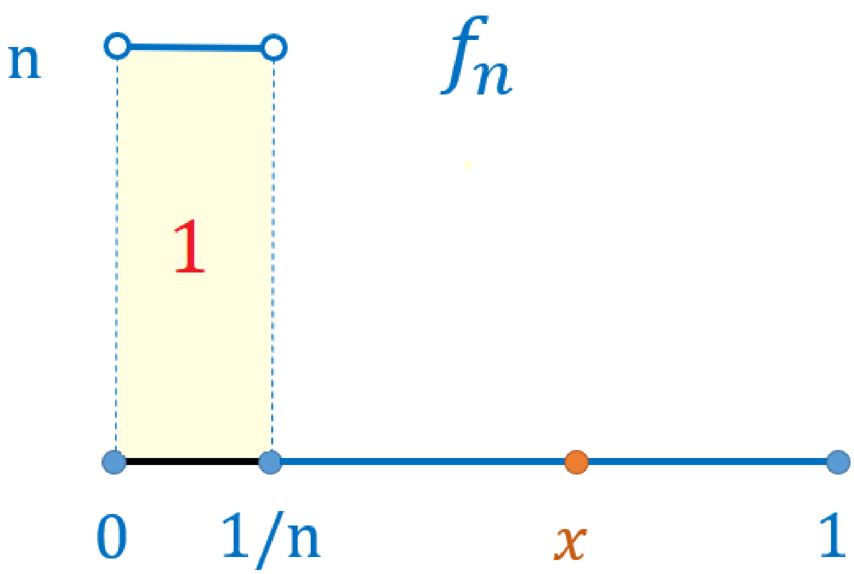


The answer is **NO**, and here's a really interesting counterexample

**Non-Example:**

Consider the following sequence of functions on $[0,1]$

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$




**Claim # 1**

For every $x$, $\lim_{n\to\infty} f_n(x) = 0$

**Why?** If $x > 0$, then if $n$ is large enough, we eventually have $\frac{1}{n} \leq x$, then by definition $f_n(x) = 0$ for all $n$ large, and so $\lim_{n\to\infty} f_n(x) = 0$. And if $x = 0$, then $f_n(0) = 0$ for all $n$ by definition, and so

$\lim_{n\to\infty} f_n(0) = 0$

**Claim # 2**

$\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 \lim_{n\to\infty} f_n$

**Why?** On the one hand, by the above, $\int_0^1 \lim_{n\to\infty} f_n(x) = \int_0^1 0 = 0$.

On the other hand, $f_n$ is a rectangle with width $\frac{1}{n}$ and height $n$, so

$$\int_0^1 f_n(x)dx = n \times \frac{1}{n} = 1$$

$$\text{And so } \lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} 1 = 1 \neq 0$$

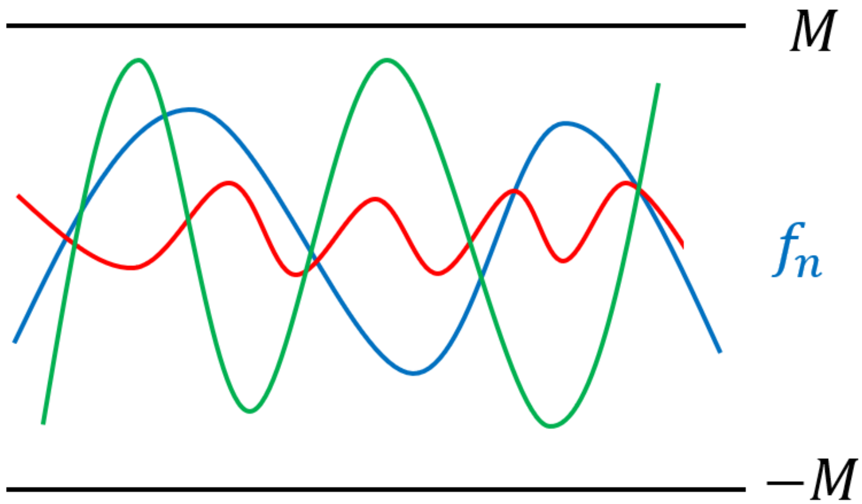
The main problem here is that the $f_n$ blows up to $\infty$ at 0. It turns out that if all the $f_n$ are bounded, then we're ok

**Bounded Convergence Theorem**

If $|f_n| \leq M$ for some $M$ independent of $n$, then

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n$$

(Provided the limits exist)




**Note:** The book incorrectly calls this the Dominated Convergence Theorem

The cool thing is that we can even replace $M$ by any integrable function $g$, like $e^{-x^2}$, provided that $g$ doesn't depend on $M$.