LECTURE 27: THE FUNDAMENTAL THEOREM OF CALCULUS

It's the most amazing of coincidences that the last topic of Advanced Calculus is about the Fundamental Theorem of Calculus! We will prove both versions.

1. FTC 2

Video: FTC 2 Proof

FTC 2

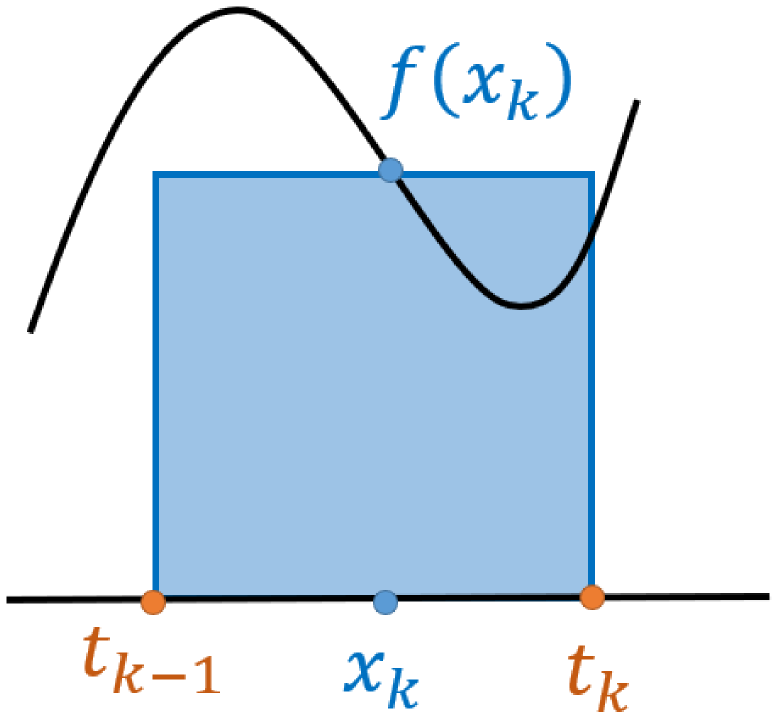
$$\int_a^b f'(x)dx = f(b) - f(b)$$

That is, the integral of a function is the difference of antiderivatives

Note: The book proves it using Darboux integration, but it's actually more elegant to do it using Riemann integrals:

Recall: Riemann Integral: Given a partition, on each sub-piece $[t_{k-1}, t_k]$ you choose an arbitrary point x_k

Date: Thursday, December 2, 2021.



And you consider the Riemann (random) sum

$$\sum_{k=1}^{n} f(x_k)\,(t_k - t_{k-1})$$

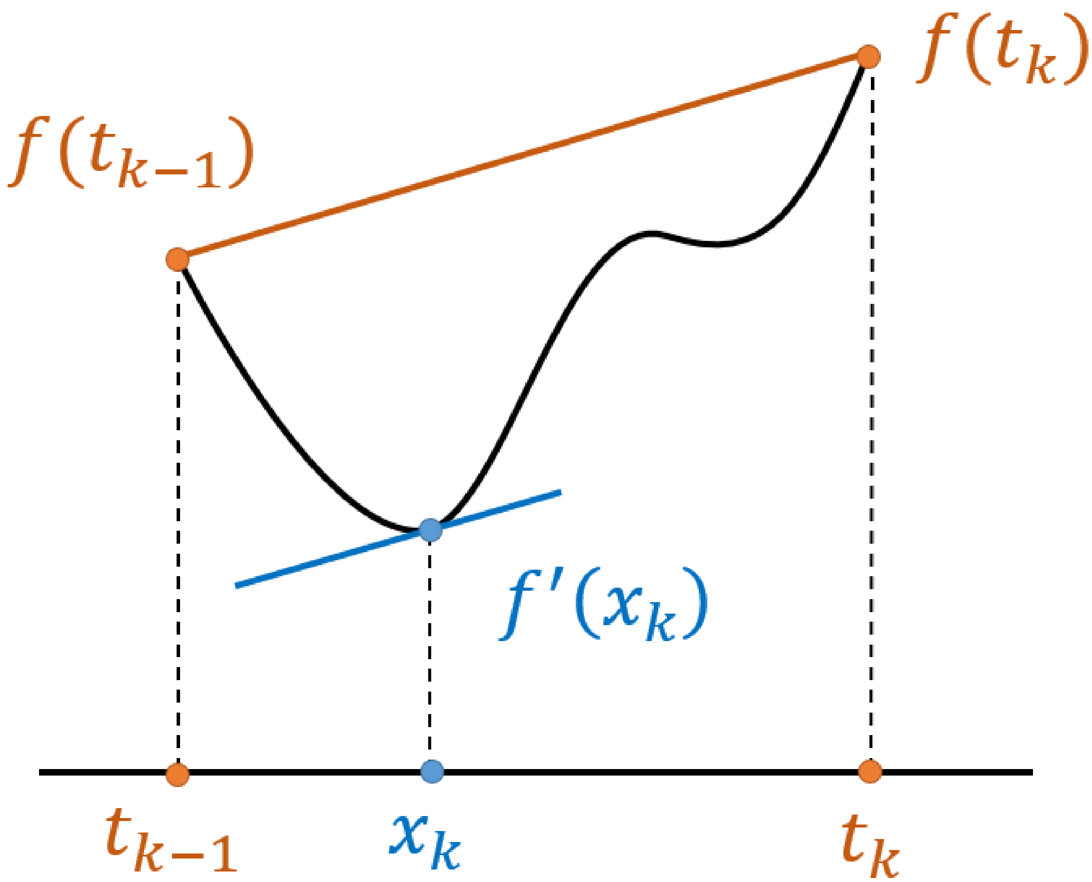
If f is integrable, then, no matter how we choose the x_k, this sum converges to $\int_a^b f(x)dx$ (as the partition P gets finer)

Proof: The idea is to choose a clever x_k that makes the Riemann sum equal to $f(b) - f(a)$

Let P be any partition of $[t_{k-1}, t_k]$

By the MVT, for every k, there is x_k in $[t_{k-1}, t_k]$ such that

$$f'(x_k) = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}}$$



This implies

$$f'(x_k)\,(t_k - t_{k-1}) = f(t_k) - f(t_{k-1})$$

With *this* choice of x_k the Riemann sum of f' becomes

$$\begin{aligned}\sum_{k=1}^{n} f'(x_k)\,(t_k - t_{k-1}) =& \sum_{k=1}^{n} f(t_k) - f(t_{k-1}) \\ =& f(t_1) - f(t_0) + f(t_2) - f(t_1) + \cdots + f(t_n) - f(t_{n-1}) \\ =& f(t_n) - f(t_0) \\ =& f(b) - f(a)\end{aligned}$$

Since this is true for *any* partition P, we get

$$\int_a^b f'(x)dx = f(b) - f(a) \quad \square$$

Note: This proof illustrates the beauty of Riemann integration: If a function is integrable, we can choose any x_k we want, it gives us the same answer (in the limit). Moreover, we didn't even have to take a limit, since the sum of $f(b) - f(a)$ for **any** partition.

2. Interlude: Integration by Parts

Before we prove the other FTC, let's prove the formula for integration by parts

Integration by Parts:

$$\int_a^b f'(x)g(x)dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)dx$$

Proof: It just follows from the product rule:

$$\begin{aligned}(f(x)g(x))' &= f'(x)g(x) + f(x)g'(x) \\ f'(x)g(x) &= (f(x)g(x))' - f(x)g'(x) \\ \int_a^b f'(x)g(x)dx &= \int_a^b (f(x)g(x))'\, dx - \int_a^b f(x)g'(x)dx \\ \int_a^b f'(x)g(x)dx &= [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)dx\end{aligned}$$

Note: The following video gives a geometric proof of the IBP formula:

Video: IBP Intuition

3. FTC 1

Video: FTC 1 Proof

Now let's cover FTC 1, the one that says the derivative of the integral is the function itself.

FTC 1

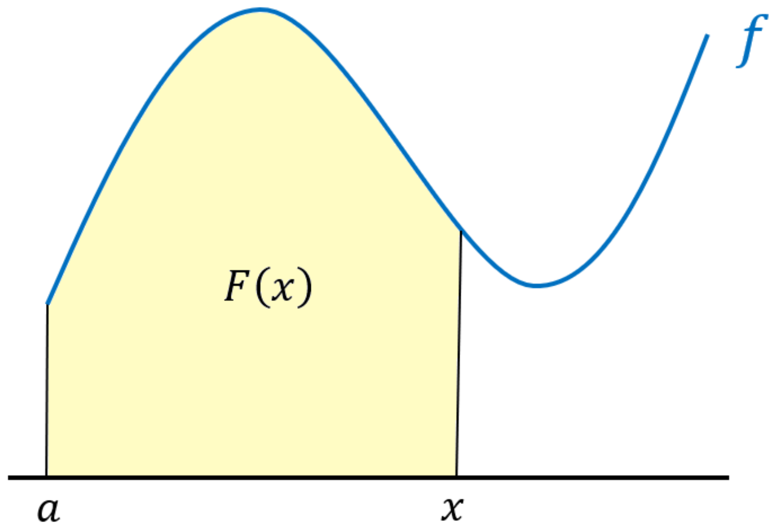
Suppose f is continuous on $[a,b]$ and define

$$F(x) = \int_a^x f(t)dt$$

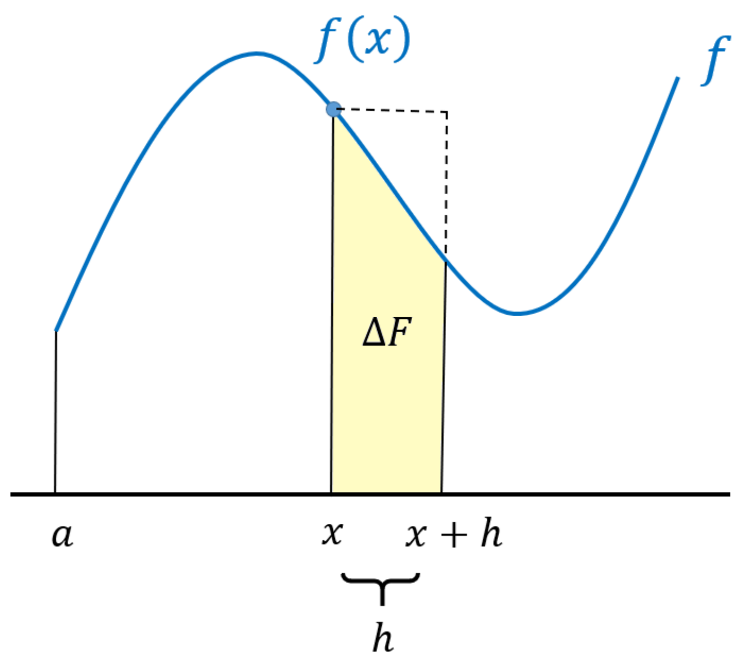
Then: $F'(x) = f(x)$

In other words, $\left(\int_a^x f(t)dt\right)' = f(x)$, the derivative of the integral is the function itself.

So F is the area under f from a to x, think water filling a tank from a to x:



Intuitively, the FTC follows from the following picture:



On the one hand, the change of F is $\Delta F = F(x+h) - F(x)$

On the other hand, the change in F is roughly equal to the area of the rectangle with base $[x, x+h]$ and height $f(x)$, which is $hf(x)$, and so:

$$F(x+h) - F(x) \approx hf(x) \Rightarrow \frac{F(x+h) - F(x)}{h} \approx f(x)$$

And taking the limit as $h \to 0$ we get $F'(x) = f(x)$

We just need to make this idea rigorous

Proof: Beautiful application of uniform continuity (!)

STEP 1: Scratchwork

Our goal is to show that $F'(x) = f(x)$, that is

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| = \left| \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} - f(x) \right| = \left| \frac{\int_x^{x+h} f(t)dt}{h} - f(x) \right|$$

Clever Observation: It would be nice if we could write $f(x)$ as an integral of the same form, but notice that:

$$f(x) = \frac{\int_x^{x+h} f(x)dt}{h}$$

Why? Since $f(x)$ doesn't depend on t, we get

$$\int_x^{x+h} f(x)dt = f(x)\int_x^{x+h} 1\,dt = f(x)(x+h-x) = f(x)h$$

And solving for $f(x)$, we get the desired identity.

Continuing, we get:

$$\begin{aligned}\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| &= \left|\frac{\int_x^{x+h} f(t)dt}{h} - \frac{\int_x^{x+h} f(x)dt}{h}\right| \\ &= \left|\frac{\int_x^{x+h} f(t) - f(x)dt}{h}\right| \\ &\leq \frac{1}{h}\int_x^{x+h} \left|f(t)-f(x)\right| dt\end{aligned}$$

(WLOG, assume $h > 0$ here)

And **this** is where continuity kicks in!

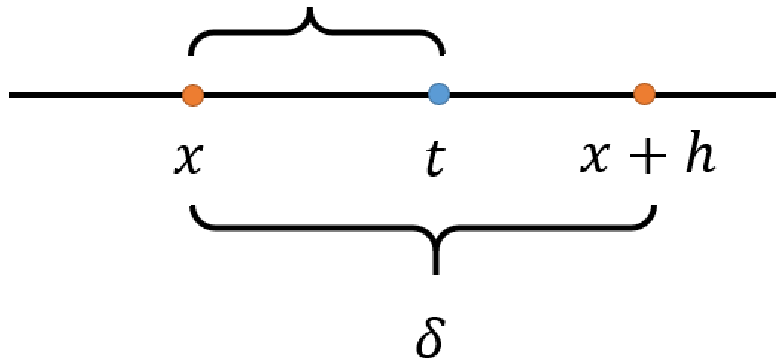
STEP 2: Actual Proof

Let $\epsilon > 0$ be given

Since f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$, and so there is $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

With the same δ, if $0 < h < \delta$ then $|f(t) - f(x)| < \epsilon$

Why? If t is in $[x, x+h]$ then $|x - t| \leq h < \delta$, and so $|f(t) - f(x)| < \epsilon$



We can continue the calculation to get

$$\begin{aligned}\left|\frac{F(x+h)-F(x)}{h}-f(x)\right| \leq & \frac{1}{h}\int_x^{x+h}|f(t)-f(x)|\,dt \\ < & \frac{1}{h}\int_x^{x+h}\epsilon\,dt \\ = & \frac{\epsilon}{h}(x+h-x) \\ = & \epsilon\left(\frac{h}{h}\right) \\ = & \epsilon\end{aligned}$$

Hence if $0 < h < \delta$, then $\left|\frac{F(x+h)-F(x)}{h}-f(x)\right| < \epsilon$

Therefore $\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x)$ that is $F'(x) = f(x)$ □

(Technically we've only shown the limit as $h \to 0^+$, but the other limit is similar)

4. FTC 1+

What is f is not continuous, but merely integrable, like $f(x) = \sin\left(\frac{1}{x}\right)$? Then F might not be differentiable, but it is still continuous.

FTC 1+

If f is integrable on $[a, b]$, then F is uniformly continuous on $[a, b]$

Analogy: If you're driving like crazy, your distance never jumps but it may have kinks or jerks.

Proof: Much easier, actually! Since f is bounded, there is $M > 0$ such that $|f(x)| \leq M$ for all x.

Let $\epsilon > 0$ be given, and let $\delta = \frac{\epsilon}{M}$, then if $|x - y| < \delta$, then WLOG, $y > x$ so

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_x^y f(t)dt \right| \\ &\leq \int_x^y |f(t)|\, dt \\ &\leq \int_x^y M dt \\ &= M(y - x) \\ &< M\left(\frac{\epsilon}{M}\right) \\ &= \epsilon \checkmark \end{aligned}$$

And so F is uniformly continuous on $[a, b]$ □

5. u-SUBSTITUTION

Finally, let's end this course with the integral version of the Chen Lu, which is $u-$sub:

u-sub:

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

This is sometimes written as $\int f(u(x)) \left(\frac{du}{dx}\right) dx = \int f(u)du$

Proof: Consider $F(u(x))$ where F is an antiderivative of f, then by the Chen Lu:

$$(F(u(x)))' = F'(u(x))u'(x) = f(u(x))u'(x)$$

And integrating this from a to b, we get:

$$\begin{aligned}
\int_a^b (F(u(x)))' dx &= \int_a^b f(u(x))u'(x)dx \\
F(u(b)) - F(u(a)) &= \int_a^b f(u(x))u'(x)dx \\
[F(x)]_{u(a)}^{u(b)} &= \int_a^b f(u(x))u'(x)dx \\
\int_{u(a)}^{u(b)} f(u)du &= \int_a^b f(u(x))u'(x)dx
\end{aligned}$$

In the last step we used that F is an antiderivative of f, so $\int f = F$

6. Two Fun Examples

Finally, to end on a high note, let me show you two cool integration tricks, starting with integrals of inverse functions:

Example 1:

$$\int_0^{\frac{1}{2}} \sin^{-1}(x)dx$$

(1) $x = \sin(u)$

(2) $dx = \cos(u)du$

(3) $0 = \sin(u) \Rightarrow u = 0$ and $\frac{1}{2} = \sin(u) \Rightarrow u = \frac{\pi}{6}$

(4)

$$\begin{aligned}
\int_0^{\frac{1}{2}} \sin^{-1}(x)dx &= \int_0^{\frac{\pi}{6}} \sin^{-1}(\sin(u))\cos(u)du \\
&= \int_0^{\frac{\pi}{6}} u\cos(u)du \\
&= [u\sin(u)]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin(u)du \qquad \text{(IBP)} \\
&= \frac{\pi}{6}\sin\left(\frac{\pi}{6}\right) - 0\sin(0) - \int_0^{\frac{\pi}{6}} \sin(u)du \\
&= \frac{\pi}{6}\left(\frac{1}{2}\right) + [\cos(u)]_0^{\frac{\pi}{6}} \\
&= \frac{\pi}{12} + \cos\left(\frac{\pi}{6}\right) - \cos(0) \\
&= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1
\end{aligned}$$

Notice that in the process we got

$$\int_0^{\frac{1}{2}} \sin^{-1}(x)dx = \frac{\pi}{6}\sin\left(\frac{\pi}{6}\right) - 0\sin(0) - \int_0^{\frac{\pi}{6}} \sin(u)du$$

Using the same technique, this can be generalized as follows:

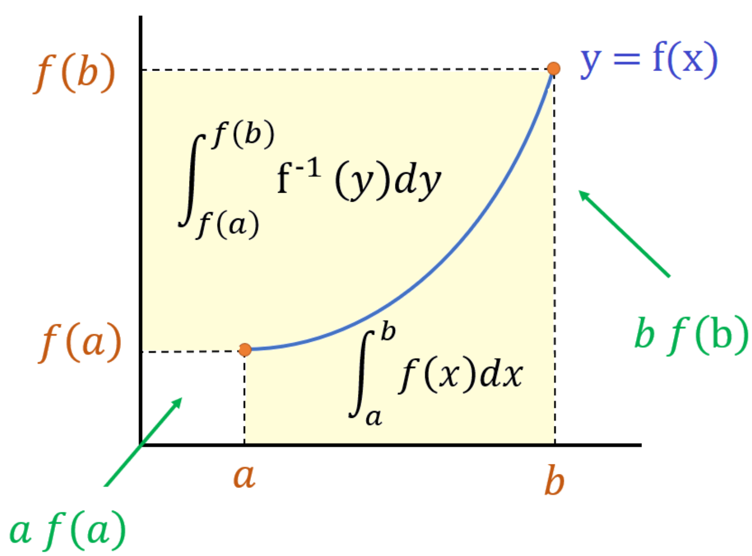
Fact

$$\int_{f(a)}^{f(b)} f^{-1}(x)dx = bf(b) - af(a) - \int_a^b f(u)du$$

Or, in other words:

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(y)dy = bf(b) - af(a)$$

This has a nice graphical interpretation: The sum of the areas under the graph of f and to the left of the graph of f (in yellow below) equals to the difference of the areas of the rectangles $bf(b)$ and $af(a)$:



Finally, let me show you one of my favorite integration tricks:

Example 2:

$$\int_0^1 x \tan^{-1}(x) dx$$

Usually you would integrate this by parts, using $v = \tan^{-1}(x)$, $dv = \frac{1}{x^2+1}$, $du = x$ and $u = \frac{x^2}{2}$, but that gives a nasty integral!

Trick: remember that you can add a constant to any antiderivative and still get an antiderivative! So instead of doing $u = \frac{x^2}{2}$, try out $u = \frac{x^2+1}{2}$, then you get:

$$\begin{aligned}\int_0^1 x \tan^{-1}(x) dx &= \left[\left(\frac{x^2+1}{2}\right)\tan^{-1}(x)\right]_0^1 - \int_0^1 \left(\frac{x^2+1}{2}\right)\left(\frac{1}{x^2+1}\right) dx \\ &= \left(\frac{2}{2}\right)\tan^{-1}(1) - \frac{1}{2}\tan^{-1}(0) - \int_0^1 \frac{1}{2} dx \\ &= \frac{\pi}{4} - \frac{1}{2}\end{aligned}$$

And with this, we are officially done with the course! Next time we'll have a semester in review ☺

The End