LECTURE 28: SPHERICAL COORDINATES (I)

Today is all about a really useful coordinate system that simplifies integrals tremendously: Spherical Coordinates.

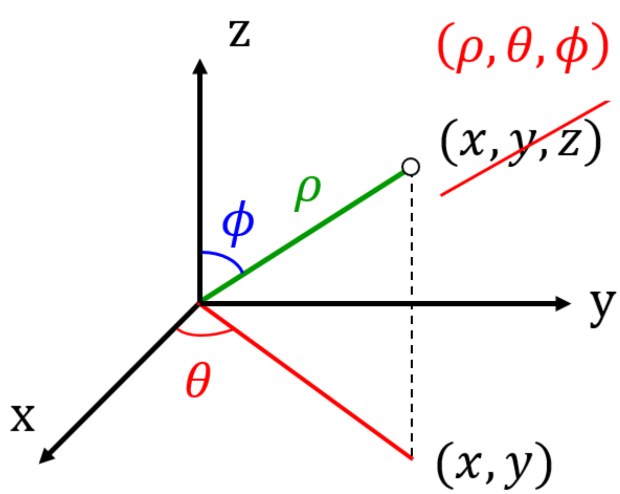
1. Spherical Coordinates

Idea: Represent (x, y, z) as (ρ, θ, ϕ), where:

(1) $\rho =$ distance from 0 to (x, y, z) (**RHO**dius)

(2) $\theta =$ angle between (x, y) and x-axis (**THO**rizontal)

(3) $\phi =$ angle between (x, y, z) and $z-$axis (**PHE**rtical)



Date: Monday, November 1, 2021.

Mnemonic: θ is 0 with a *horizontal* line (so horizontal) and ϕ is 0 with a *vertical* line (so vertical)

Constraints:

$$\rho \geq 0$$
$$0 \leq \theta \leq 2\pi$$
$$⚠ 0 \leq \phi \leq \pi$$

(The reason for the last one is because we can't go more south than the South pole)

Most important property:

$$x^2 + y^2 + z^2 = \rho^2$$

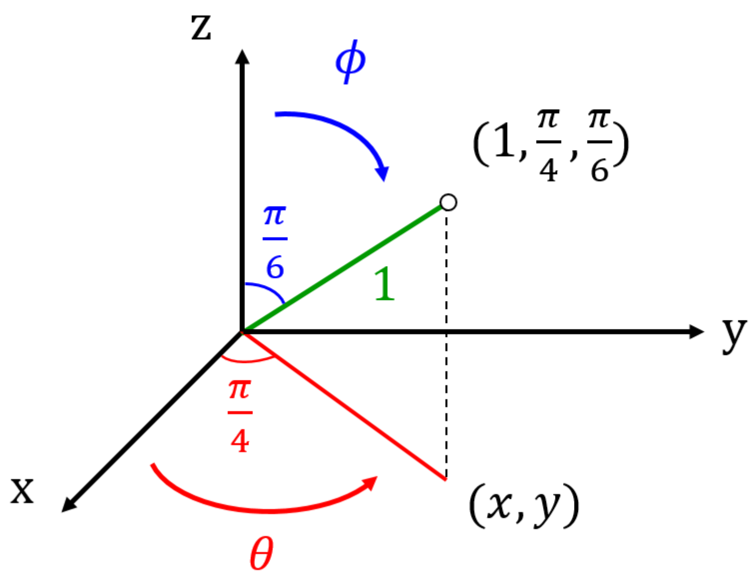
So spherical is really the natural analog of polar but in 3 dimensions!

Example 1:

Plot the following points:

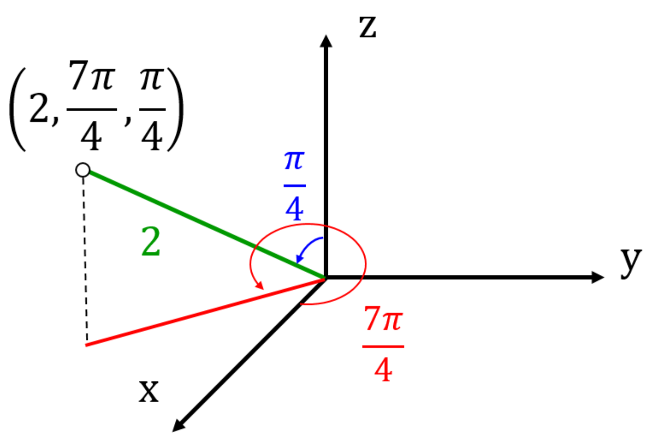
(a) $\left(1, \frac{\pi}{4}, \frac{\pi}{6}\right)$

Think like a robot! You move horizontally right by $\frac{\pi}{4}$ (from the $x-$axis) and then you move vertically down by $\frac{\pi}{6}$ (from the $z-$axis)



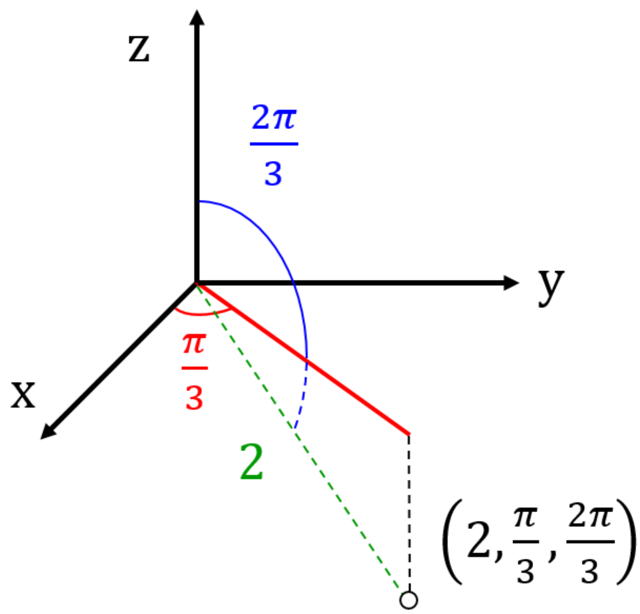
(b) $\left(2, \frac{7\pi}{4}, \frac{\pi}{4}\right)$

(Remember $\frac{7\pi}{4}$ is the same as $\frac{-\pi}{4}$)



(c) $\left(2, \frac{\pi}{3}, \frac{2\pi}{3}\right)$

Here you're moving *below* the $xy-$plane



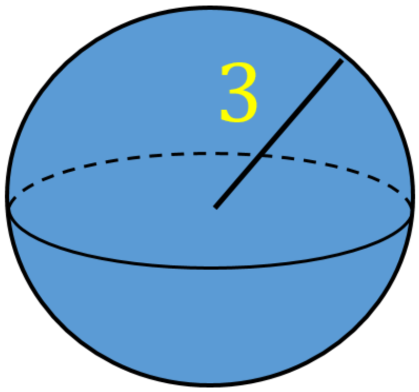
Just like for cylindrical coordinates, this is useful because a lot of familiar objects can be written elegantly in terms of spherical coordinates.

Example 2:

Sketch the following surfaces

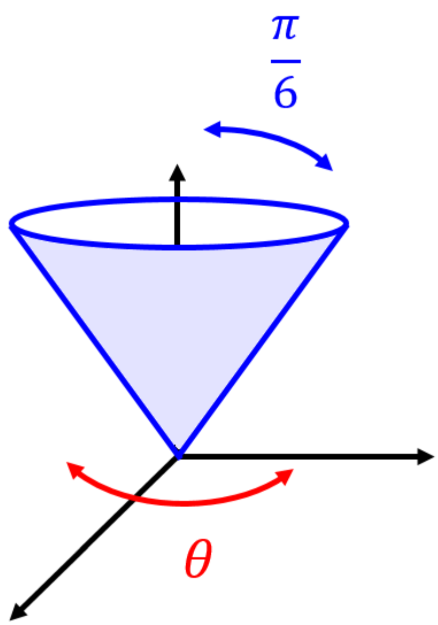
(a) $\rho = 3$

$$\sqrt{x^2+y^2+z^2} = 3 \Rightarrow x^2+y^2+z^2 = 9 \text{ Sphere}$$



(b) $\phi = \frac{\pi}{6}$

It's an upper-cone with vertical angle $\phi = \frac{\pi}{6} = 30^\circ$



Note: The lower cone would be $\phi = \frac{5\pi}{6} = 150^\circ$

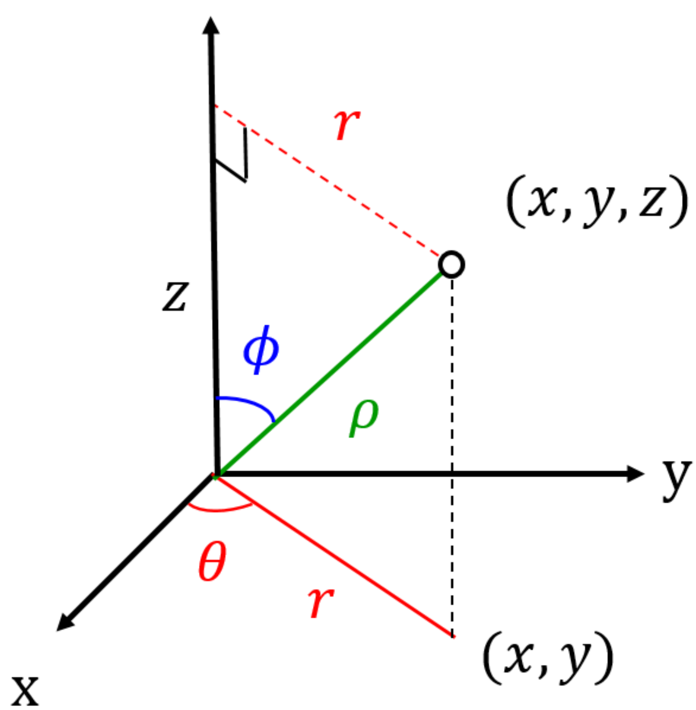
2. Derivation of Spherical Coordinates

Video: Derivation of Spherical Coordinates

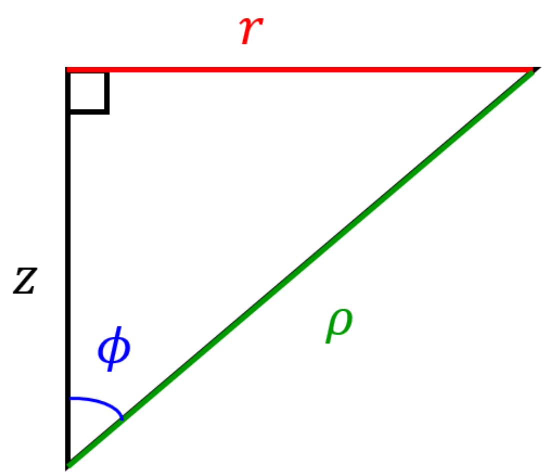
Goal: Find equations for x, y, z in terms of ρ, θ, ϕ (similar to $x = r\cos(\theta)$ for polar coordinates)

STEP 1: Picture:

Here r is the distance between O and (x, y), just like for cylindrical coordinates



STEP 2: Focus on the following triangle:



By SOHCAHTOA, we have:

$$\cos(\phi) = \frac{z}{\rho} \Rightarrow z = \rho\cos(\phi)$$

And also:

$$\sin(\phi) = \frac{r}{\rho} \Rightarrow r = \rho\sin(\phi)$$

STEP 3: The rest is just polar coordinates and the formula for r above:

$$x = r\cos(\theta) \Rightarrow x = \rho\sin(\phi)\cos(\theta)$$
$$y = r\sin(\theta) \Rightarrow y = \rho\sin(\phi)\sin(\theta)$$

Spherical Coordinates:

$$\begin{cases} x = \rho\sin(\phi)\cos(\theta) \\ y = \rho\sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \end{cases}$$

Mnemonic: For $z = \rho\cos(\phi)$, use the $z - \phi - \rho$ triangle above and for x and y, use $x = r\cos(\theta)$ and $y = r\sin(\theta)$

3. Integrals with Spherical Coordinates

Spherical coordinates are literally the Bazooka of math; they allow us to simplify complicated integrals like crazy!

Rule of Thumb

Spherical is great for spheres and cones.

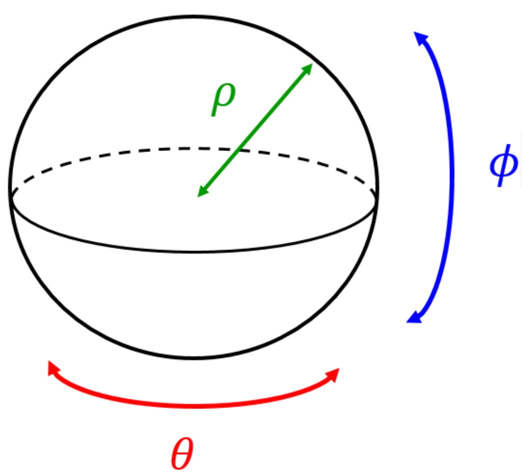
Example 3:

Find the volume of a ball of radius R.

STEP 1:

$$V = \int\int\int_E 1\, dxdydz \quad E = \text{ Ball of radius } R$$

STEP 2: Picture:



STEP 3: Inequalities: No restrictions on θ and ϕ, so it's

$$\begin{cases} 0 \leq \rho \leq R \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{cases}$$

STEP 4: Integrate

Fact:

$$\int\int\int_E f(x, y, z)\, dxdydz = \int\int\int_E f(\rho, \theta, \phi)\rho^2 \sin(\phi)\, d\rho d\theta d\phi$$

$$\begin{aligned}
&\int\int\int_E 1\, dxdydz \\
=&\int_0^{\pi}\int_0^{2\pi}\int_0^R \rho^2 \sin(\phi)\, d\rho d\theta d\phi \\
=&\left(\int_0^R \rho^2 d\rho\right)\left(\int_0^{\pi} \sin(\phi) d\phi\right)\left(\int_0^{2\pi} 1 d\theta\right) \\
=&\frac{R^3}{3}(2)(2\pi) \\
=&\frac{4}{3}\pi R^3
\end{aligned}$$

Note: Where does the $\rho^2 \sin(\phi)$ come from? Roughly speaking, before we had $r\, drd\theta$ but now we have $\rho r\, drd\theta d\phi$, and

$$\rho r = \rho(\rho \sin(\phi)) = \rho^2 \sin(\phi)$$

If you want a more geometric explanation, please see the *optional* appendix at the end.

4. Ice Cream Cone (again!)

Video: Spherical Coordinates and Ice Cream Cones

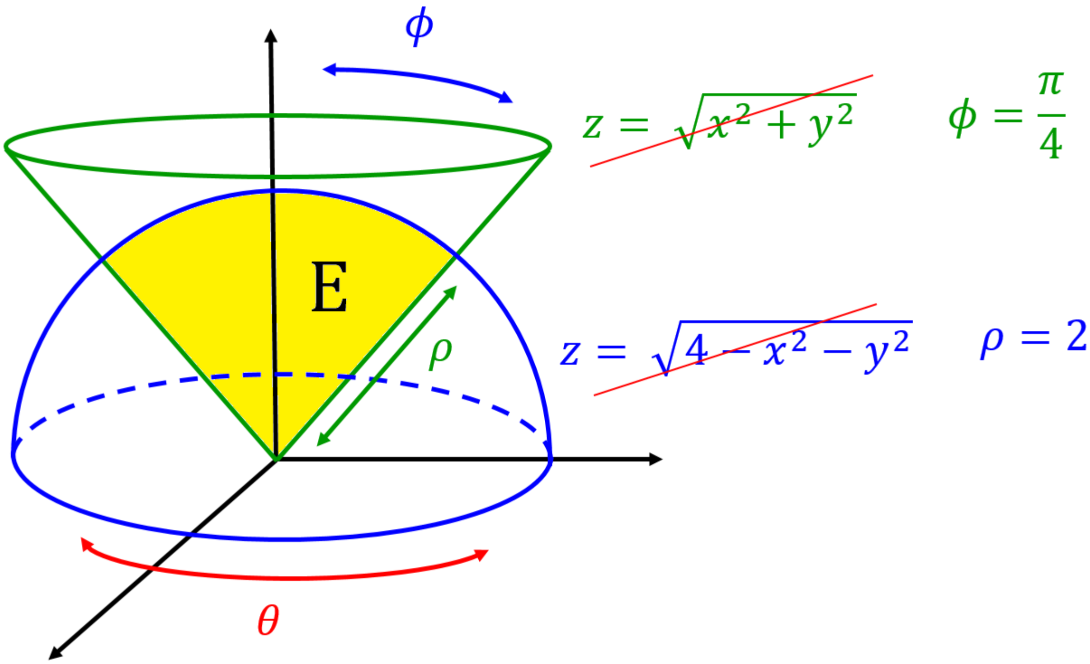
To really see how amazing spherical coordinates are, another ice cream cone example, to see how much easier this becomes!

Example 4:

Calculate the following integral where E is the region above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $z = \sqrt{4-x^2-y^2}$

$$\int\int\int_E z \, dxdydz$$

STEP 1: Picture:



STEP 2: Inequalities:

$$z = \sqrt{x^2+y^2} = r \Rightarrow \rho\cos(\phi) = \rho\sin(\phi) \Rightarrow \cos(\phi) = \sin(\phi) \Rightarrow \phi = \frac{\pi}{4}$$

$$\begin{cases} 0 \leq \rho \leq 2 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \frac{\pi}{4} \end{cases}$$

STEP 3: Integrate:

$$\begin{aligned}
&\int\int\int_E z\, dxdydz \\
=&\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^2 \rho\cos(\phi)\,\rho^2\sin(\phi)\, d\rho d\theta d\phi \\
=&2\pi\left(\int_0^2 \rho^3 d\rho\right)\left(\int_0^{\frac{\pi}{4}}\sin(\phi)\cos(\phi)d\phi\right) \\
&\left(u=\sin(\phi), du=\cos(\phi)d\phi, u(0)=0, u\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\right) \\
=&2\pi\left(\frac{2^4}{4}\right)\left(\int_0^{\frac{1}{\sqrt{2}}} udu\right) \\
=&2\pi(4)\left[\frac{u^2}{2}\right]_0^{\frac{1}{\sqrt{2}}} \\
=&8\pi\left(\frac{1}{4}\right) \\
=&2\pi
\end{aligned}$$

Note: For more practice, check out the following video:

Video: Volume of an Ice Cream Cone

5. More Integration Practice

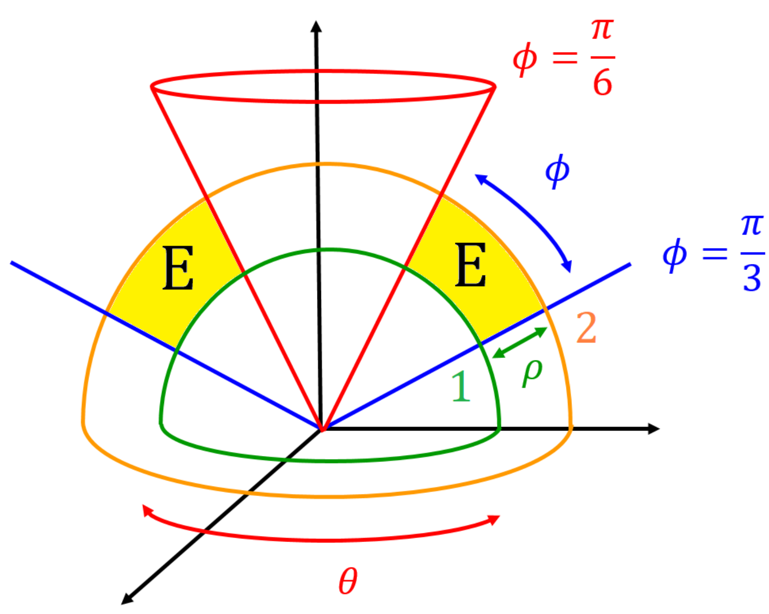
Example 5: (extra practice)

Calculate the following integral where E is the solid between $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, $\phi = \frac{\pi}{6}$ $\phi = \frac{\pi}{3}$

$$\int\int\int_E \sqrt{x^2 + y^2}\, dxdydz$$

STEP 1: Picture:

$$x^2 + y^2 + z^2 = 1 \Rightarrow \rho = 1$$
$$x^2 + y^2 + z^2 = 4 \Rightarrow \rho = 2$$
$$\phi = \frac{\pi}{6} \text{ and } \phi = \frac{\pi}{3} \text{ are cones}$$



STEP 2: Inequalities:

$$\begin{cases} 1 \leq \rho \leq 2 \\ 0 \leq \theta \leq 2\pi \\ \frac{\pi}{6} \leq \phi \leq \frac{\pi}{3} \end{cases}$$

STEP 3: Integrate:

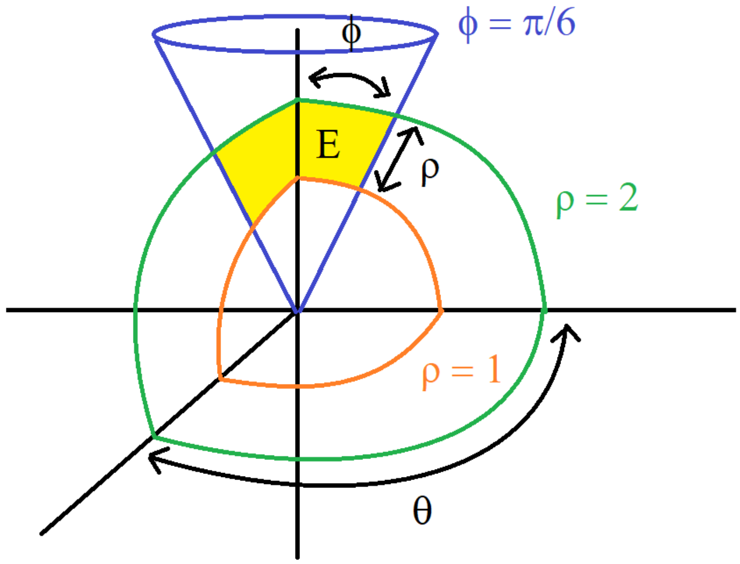
$$\begin{aligned}
&\int\int\int_E \sqrt{x^2+y^2}\, dxdydz \\
=&\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_0^{2\pi} \int_1^2 \rho \sin(\phi)\, \rho^2 \sin(\phi)\, d\rho d\theta d\phi \\
=&2\pi \left(\int_1^2 \rho^3 d\rho \right) \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2(\phi) d\phi \right) \\
=&2\pi \left[\frac{\rho^4}{4} \right]_1^2 \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} - \frac{1}{2}\cos(2\phi) d\phi \right) \\
=&2\pi \left(\frac{15}{4} \right) \left[\frac{\phi}{2} - \frac{1}{4}\sin(2\phi) \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
=&\frac{15\pi}{2} \left(\frac{\pi}{6} - \frac{\pi}{12} - \frac{1}{4}\sin\left(\frac{2\pi}{3}\right) + \frac{1}{4}\sin\left(\frac{\pi}{3}\right) \right) \\
=&\frac{15\pi}{2} \left(\frac{\pi}{12} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) \right) \\
=&\frac{15\pi}{2} \left(\frac{\pi}{12} \right) \\
=&\frac{5\pi^2}{8}
\end{aligned}$$

Example 6 (extra practice):

Calculate the following integral where E is the solid between $\rho = 1, \rho = 2$, above $\phi = \frac{\pi}{6}$, in the first octant.

$$\int\int\int_E \sqrt{x^2 + y^2 + z^2}\, dxdydz$$

STEP 1: Picture:



STEP 2: Inequalities:

$$\begin{cases} 1 \leq \rho \leq 2 \\ 0 \leq \theta \leq \dfrac{\pi}{2} \\ 0 \leq \phi \leq \dfrac{\pi}{6} \end{cases}$$

STEP 3: Integrate:

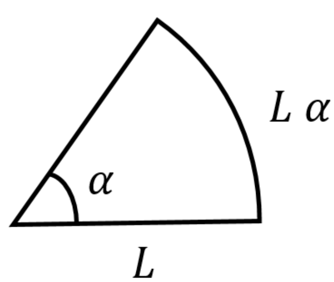
$$
\begin{aligned}
&\int\int\int_E \sqrt{x^2+y^2+z^2}\, dxdydz \\
=&\int_0^{\frac{\pi}{6}}\int_0^{\frac{\pi}{2}}\int_1^2 \rho\, \rho^2\sin(\phi)\, d\rho d\theta d\phi \\
=&\frac{\pi}{2}\left(\int_1^2 \rho^3 d\rho\right)\left(\int_0^{\frac{\pi}{6}}\sin(\phi)d\phi\right) \\
=&\frac{\pi}{2}\left[\frac{\rho^4}{4}\right]_1^2 \left[-\cos(\phi)\right]_0^{\frac{\pi}{6}} \\
=&\left(\frac{\pi}{2}\right)\left(\frac{15}{4}\right)\left[-\frac{\sqrt{3}}{2}+1\right] \\
=&\frac{15\pi}{16}\left(2-\sqrt{3}\right)
\end{aligned}
$$

6. Optional Appendix: $\rho^2\sin(\phi)$

Question: Where does $\rho^2\sin(\phi)$ come from?

Recall:

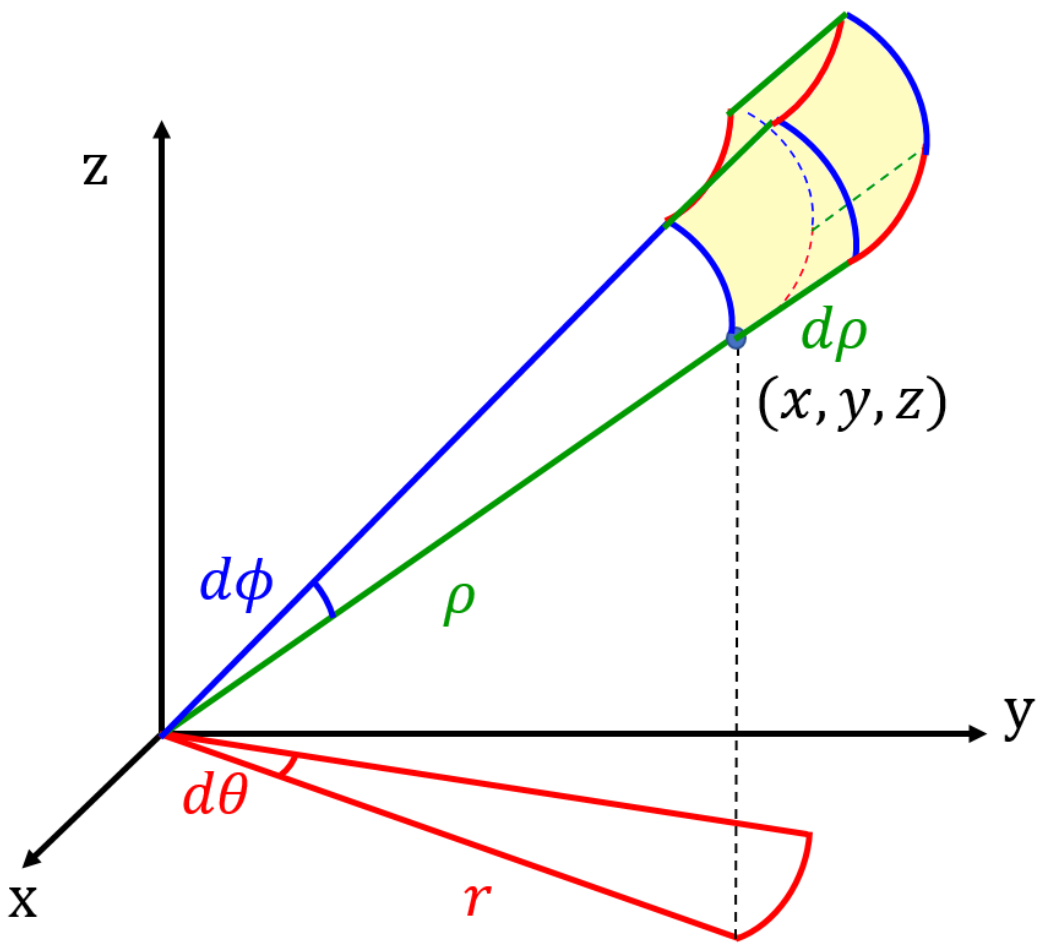
The length of an arc of radius L and angle α is $L\alpha$



This follows from proportionality: An angle of 2π (a full circle) corresponds to $2\pi L$, hence an angle of α corresponds to αL.

Now fix a point (x, y, z) and move around that point a little bit by changing ρ, θ, ϕ. If you do that, then in spherical coordinates you get a little wedge, as in the following picture:

Picture:

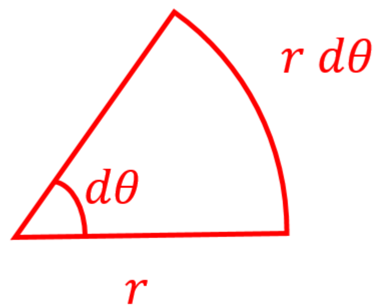


The volume of that wedge is approximately:

$$\text{Volume} \approx \text{Length} \times \text{Width} \times \text{Height}$$

(1) Length $= d\rho$ (Small change in the radius)

(2) Width $= r d\theta = \rho \sin(\phi) d\theta$



(3) Height $= \rho d\theta$ (because arclength of length ρ and angle $d\phi$)

Therefore:

$$\text{Volume} \approx (d\rho)(\rho \sin(\phi) d\theta)(\rho d\phi) = \rho^2 \sin(\phi) d\rho d\theta d\phi$$