LECTURE 28: FINAL EXAM REVIEW

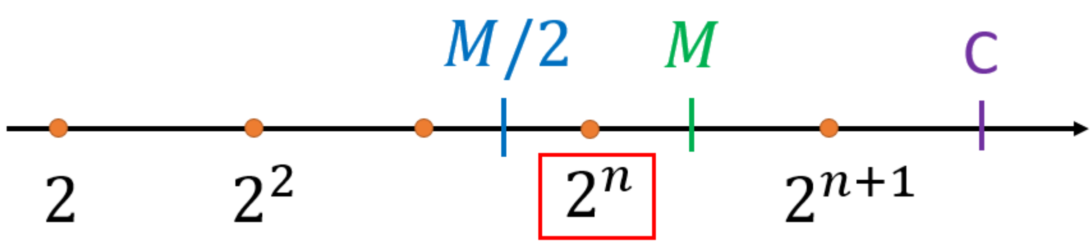
There's a saying in German that says "Everything has an end, except for a sausage, which has two."[1] And with this, I would like to welcome you to the final lecture of this course.

Today: Our year in review, with some sample final exam questions

1. It all started with Archimedes

Example 1: (Mandarin, 1 = Yi)

Show that for all $C > 0$, there is some n such that $2^n > C$



Suppose not, that is there is $C > 0$ such that for all n we have $2^n \leq C$

Consider the set $S = \{2^n \mid n \in \mathbb{N}\}$

Then by assumption S is bounded above by C, and therefore has a least upper bound $M = \sup(S)$

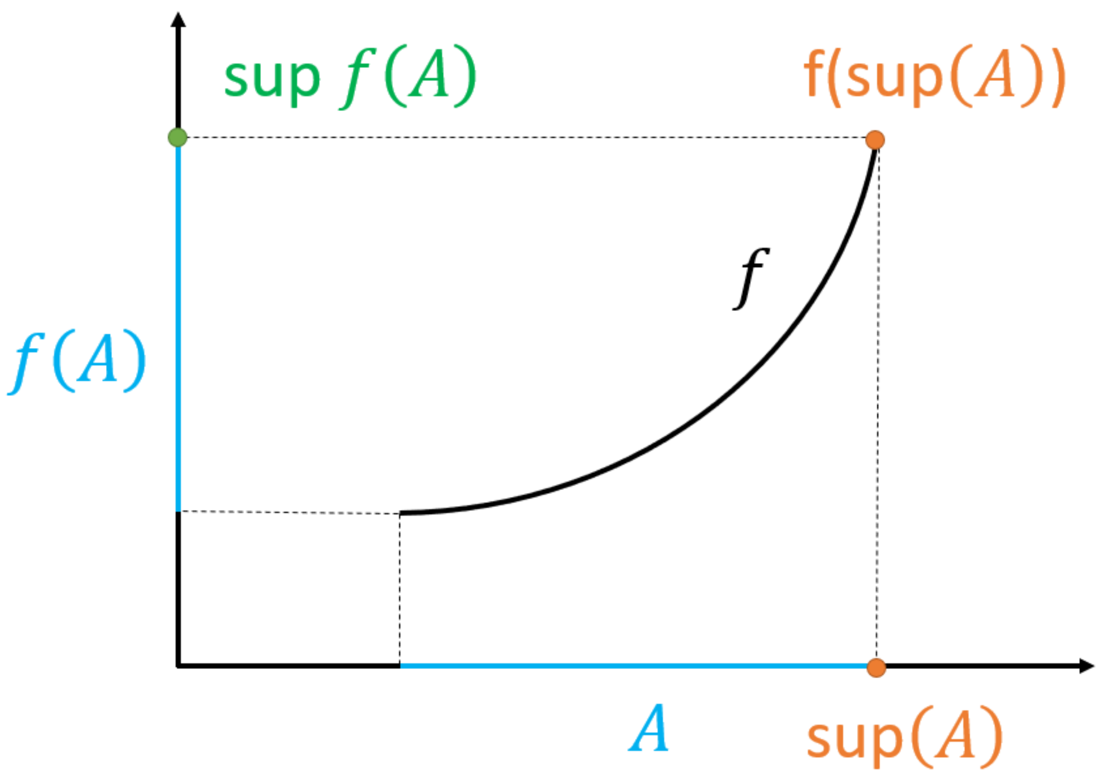
Date: Tuesday, December 7, 2021.

[1]Alles hat ein Ende, nur die Wurst, die hat zwei

Consider $\frac{M}{2} < M = \sup(S)$, so by definition of sup, there is n such that $2^n > \frac{M}{2}$, but then $2^{n+1} > M$ which is a contradiction since $2^{n+1} \in S$ but $M = \sup(S) \Rightarrow\Leftarrow$ □

Example 2: (Spanish, 2 = Dos)

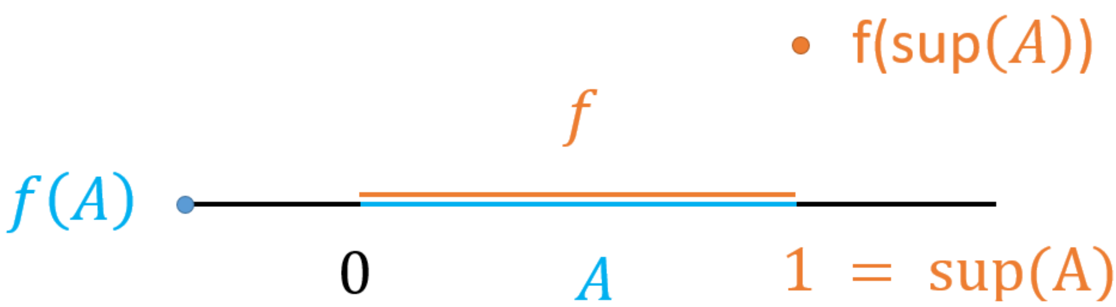
Let f be a non-decreasing function on $\mathbb{R}$ and let A be a bounded subset of $\mathbb{R}$. Show that $\sup f(A) \leq f(\sup(A))$ and find f such that $\sup f(A) < f(\sup(A))$



Notice for all $a \in A$, we have $a \leq \sup(A)$ and so since f is non-decreasing, we get $f(a) \leq f(\sup(A))$. Taking the sup over $a \in A$, we get $\sup f(A) \leq f(\sup(A))$

As a counterexample, let $A = (0, 1)$ and consider:

$$f(x) = \begin{cases} 0 & \text{on } (0,1) \\ 1 & \text{if } x = 1 \end{cases}$$



Then $f(A) = f((0,1)) = \{0\}$ so $\sup f(A) = 0$ but $f(\sup(A)) = f(1) = 1$, so $\sup f(A) < f(\sup(A))$

(It turns out that if f is continuous at $\sup(A)$, then we have equality)

2. lim inf AND lim sup

Example 3: (English, 3 = three)

Suppose $\frac{1}{2} \leq s_n \leq 2$ for all n, show that

$$\liminf_{n\to\infty} \frac{1}{s_n} = \frac{1}{\limsup_{n\to\infty} s_n}$$

In order to deal with lim inf or lim sup, the trick is to either use the helper sequences u_N and v_N, or to use subsequences

The proof below is similar in spirit to the limsup product rule

STEP 1: Let (s_{n_k}) be a subsequence of (s_n) converging to $s =: \limsup_{n\to\infty} s_n \neq 0$.

Then $\frac{1}{s_{n_k}} \to \frac{1}{s}$, so $\frac{1}{s}$ is a limit point of $\frac{1}{s_n}$

But since $\liminf_{n\to\infty} \frac{1}{s_n}$ is the smallest possible limit point, we get

$$\liminf_{n\to\infty} \frac{1}{s_n} \leq \frac{1}{s} = \frac{1}{\limsup_{n\to\infty} s_n}$$

STEP 2: Similarly, let $\left(\frac{1}{s_{n_k}}\right)$ be a (possibly different) subsequence of $\left(\frac{1}{s_n}\right)$ converging to $t =: \liminf \frac{1}{s_n} \neq 0$

Then $s_{n_k} \to \frac{1}{t}$, and since $\limsup_{n\to\infty} s_n$ is the largest possible limit point, we get

$$\limsup_{n\to\infty} s_n \geq \frac{1}{t} = \frac{1}{\liminf_{n\to\infty} \frac{1}{s_n}}$$

$$\text{That is: } \liminf_{n\to\infty} \frac{1}{s_n} \geq \frac{1}{\limsup_{n\to\infty} s_n}$$

Combining both steps, we get the desired result □

3. Why so Series?

Example 4: (Bengali, 4 = Tchar)

Suppose (s_n) is a sequence such that, for all $n \geq 1$, we have

$$|s_{n+1} - s_n| \leq \frac{1}{n^2}$$

Show that (s_n) converges

Hint: Show that (s_n) is Cauchy. For this apply the Cauchy criterion to $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Let $\epsilon > 0$ be given. Then since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (by the Integral Test) it satisfies the Cauchy criterion, and hence there is N such that if $n \geq m > N$, then

$$\left| \sum_{k=m}^{n} \frac{1}{k^2} \right| < \epsilon$$

But then, with the same N, if $m, n > N$, WLOG, $n > m$, and therefore

$$\begin{aligned} |s_m - s_n| &= |s_m - s_{m+1} + s_{m+1} - s_{m+2} + \cdots + s_{n-1} - s_n| \\ &\leq |s_m - s_{m+1}| + |s_{m+1} - s_{m+2}| + \cdots + |s_{n-1} - s_n| \\ &\leq \frac{1}{m^2} + \frac{1}{(m+1)^2} + \cdots + \frac{1}{(n-1)^2} \\ &= \sum_{k=m}^{n-1} \frac{1}{k^2} \\ &< \sum_{k=m}^{n} \frac{1}{k^2} \\ &< \epsilon \checkmark \end{aligned}$$

Hence (s_n) is Cauchy, and therefore (s_n) converges $\square$

4. Fun Continuity Problem

Example 5: (Hindi, 5 = Panj)

Suppose f is a continuous function on $(-1, 1)$ that satisfies $f(2x) = f(x)$ for all x. Show that f is constant

The trick is to notice that

$$f(x) = f\left(2\left(\frac{x}{2}\right)\right) = f\left(\frac{x}{2}\right) = f\left(2\left(\frac{x}{4}\right)\right) = f\left(\frac{x}{4}\right) = f\left(\frac{x}{8}\right)\cdots$$

And, more generally, by induction you can show that for all n, $f\left(\frac{x}{2^n}\right) = f(x)$

Now if x is fixed, consider the sequence $s_n = \frac{x}{2^n} \to 0$. Then $s_n \to 0$, and since f is continuous, we get $f(s_n) \to f(0)$

On the other hand, $f(s_n) = f\left(\frac{x}{2^n}\right) = f(x)$ for all n, so $f(s_n) \to f(x)$

Therefore, comparing limits, we have $f(x) = f(0)$ $\square$

5. The Meme of the Course

Example 6: (Portuguese, 6 = Seis)

Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose there is a sequence (s_n) in $[a, b]$ such that $0 \leq f(s_n) \leq \frac{1}{n}$ for all n Show that there is some $x \in [a, b]$ with $f(x) = 0$

We have to be careful because don't know whether (s_n) converges!

Since (s_n) is a sequence in $[a,b]$, (s_n) is bounded, and therefore, by the Bolzano-Weierstraß Theorem, (s_n) has a convergent subsequence (s_{n_k}) that converges to some $x \in [a,b]$.

Since $s_{n_k} \to x$ and f is continuous, $f(s_{n_k}) \to f(x)$.

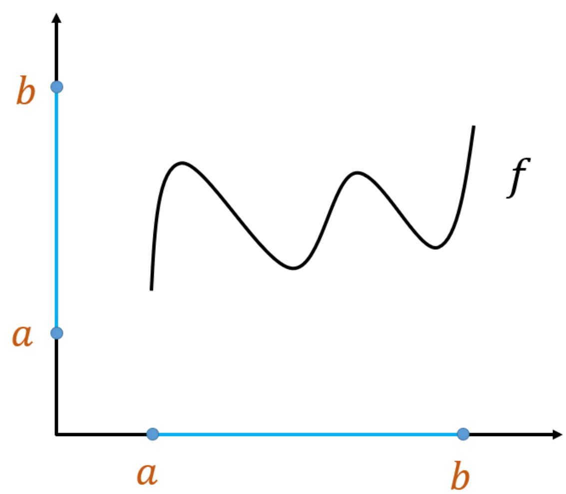
But, on the other hand, since $0 \leq f(s_n) \leq \frac{1}{n}$, we have $0 \leq f(s_{n_k}) \leq \frac{1}{n_k}$, and $\frac{1}{n_k} \to 0$, by the squeeze theorem, we have $f(s_{n_k}) \to 0$.

Combining $f(s_{n_k}) \to f(x)$ and $f(s_{n_k}) \to 0$, we get $f(x) = 0$ □

6. I Value our Friendship

Example 7: (Russian, 7 = Siem)

Suppose $0 < a < b$ and let $f : [a,b] \to [a,b]$ be a continuous function, show that there is c in (a,b) such that $cf(c) = ab$



Consider $g(x) = xf(x)$, which is continuous

$g(a) = af(a) \leq a(b) = ab$ (since $f(x) \leq b$ for all x)

$g(b) = bf(b) \geq b(a) = ab$ (since $f(x) \geq a$ for all x)

Therefore by the IVT, there is c in (a, b) such that $g(c) = ab$, that is $cf(c) = ab$

Example 8: (Japanese, 8 = Hadshee)

Suppose f is a function with $\int_0^1 f(x)dx = \frac{1}{3}$, show there is $c \in (0, 1)$ such that $f(c) = c^2$

Let $g(x) = f(x) - x^2$, then by the MVT for integrals, we have

$$\frac{1}{1-0}\int_0^1 g(x)dx = g(c)$$

$$\int_0^1 g(x)dx = \int_0^1 f(x) - x^2 dx = \int_0^1 f(x)dx - \int_0^1 x^2 dx = \frac{1}{3} - \frac{1}{3} = 0$$

Therefore the above becomes $0 = g(c)$ so $f(c) - c^2 = 0$ so $f(c) = c^2$ $\square$

7. Uniform Continuity

Example 9: (German, 9 = Neun)

Show that if f is uniformly continuous on (a, b), then f is bounded

Suppose not, then for every n there is s_n such that $|f(s_n)| \geq n$. Since (s_n) is bounded, by Bolzano-Weierstraß, (s_n) has a convergent subsequence (s_{n_k}). In particular, (s_{n_k}) is Cauchy

So since f is uniformly continuous, $f(s_{n_k})$ is Cauchy as well, so it is bounded, but this contradicts that, by assumption, $|f(s_{n_k})| \geq n_k \to \infty$ $\Rightarrow\Leftarrow$ □

8. A Chill Limit

Example 10: (Cantonese, 10 = Tsa)

Use the definition of a limit to show

$$\lim_{x \to 3^-} \frac{1}{(x-3)^3} = -\infty$$

For limits, I could ask you any possible scenario, infinite limits, limits at infinity, one-sided limits, etc.

Here we want to show that for every $M < 0$ there is δ such that if $0 < 3 - x < \delta$ then $\frac{1}{(x-3)^3} < M$

STEP 1: Scratchwork

$$\frac{1}{(x-3)^5} < M \Rightarrow (x-3)^3 > \frac{1}{M} \Rightarrow x - 3 > \frac{1}{\sqrt[3]{M}} \Rightarrow 3 - x < \frac{-1}{\sqrt[3]{M}}$$

STEP 2: Actual Proof:

Let $M < 0$ be given, let $\delta = \frac{-1}{\sqrt[3]{M}} > 0$, then if $0 < 3 - x < \delta$, then

$$3 - x < \frac{-1}{\sqrt[3]{M}} \Rightarrow x - 3 > \frac{1}{\sqrt[3]{M}} \Rightarrow (x-3)^3 > \frac{1}{M} \Rightarrow \frac{1}{(x-3)^3} < M$$

Which is what we wanted to show □

9. Derivatives

Example 11: (Javanese, 11 = Sevelas)

Show that if f is continuous at 0, then $g(x) = xf(x)$ is differentiable at 0

$$\begin{aligned}\lim_{x\to 0} \frac{g(x)-g(0)}{x} &= \lim_{x\to 0} \frac{xf(x)-0f(0)}{x} \\ &= \lim_{x\to 0} \frac{xf(x)}{x} \\ &= \lim_{x\to 0} f(x) \\ &= f(0)\end{aligned}$$

In particular the limit exists, and so g is differentiable at 0 with $g'(0) = f(0)$

10. Integrals

Example 12: (Korean, 12 = Yeldo)

Show that if f is bounded and integrable on $[a, b]$, then f^3 is integrable

Hint: Try to estimate $(f(x))^3 - (f(y))^3$ first

STEP 1: Let's first prove the hint:

$$\begin{aligned}\left|(f(x))^3-(f(y))^3\right| &= \left|(f(x)-f(y))\left((f(x))^2+f(x)f(y)+(f(y))^2\right)\right| \\ &\leq |f(x)-f(y)|\left(|f(x)|^2+|f(x)|\,|f(y)|+|f(y)|^2\right) \\ &\leq |f(x)-f(y)|\left|B^2+BB+B^2\right| \\ &= 3B^2\,|f(x)-f(y)|\end{aligned}$$

Where B is an upper bound of f

Remember that for integral problems, we have to "build" things up, starting from the sub-interval and then taking sums:

STEP 2: $M(f^3,[t_{k-1},t_k])$

On each sub-interval $[t_{k-1},t_k]$, by taking the sup over x and the inf over y in the identity

$$\left|(f(x))^3-(f(y))^3\right| \leq 3B^2\,|f(x)-f(y)|$$

We obtain:

$$M(f^3,[t_{k-1},t_k])-m(f^3,[t_{k-1},t_k]) \leq 3B^2\left(M(f,[t_{k-1},t_k])-m(f,[t_{k-1},t_k])\right)$$

STEP 3: $U(f^3,P)$

And taking the sum over k, this implies

$$U(f^3,P)-L(f^3,P) \leq 3B^2\left(U(f,P)-L(f,P)\right)$$

STEP 4: Cauchy Criterion

Since we don't know what the integral of f is, it's best to use the **Cauchy criterion**.

Let $\epsilon > 0$ be given. Then since f is integrable, there is a partition P such that $U(f, P) - L(f, P) < \frac{\epsilon}{3B^2}$.

With the same partition P, we get

$$U(f^3, P) - L(f^3, P) \leq 3B^2 \left(U(f, P) - L(f, P)\right) < 3B^2 \left(\frac{\epsilon}{3B^2}\right) = \epsilon \checkmark$$

Hence f^3 is integrable on $[a, b]$ $\square$

Alright!!! This is officially the end of Math 409 and your analysis adventure! Thank you for flying Peyam Airlines, it's been a pleasure having you on board, and I wish you a safe onward journey!

The End