LECTURE 29: SPHERICAL (II) + THE JACOBIAN (I)

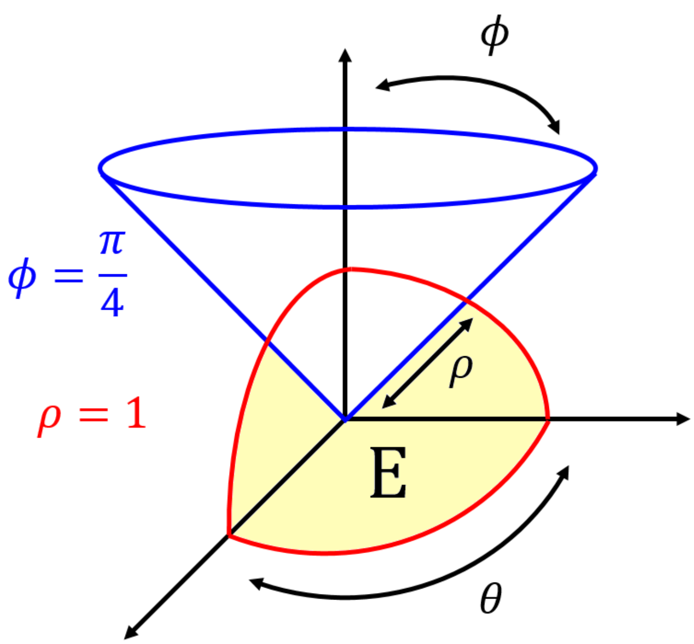
1. More Spherical Practice

Example 1:

$$\int\int\int_E x \, dxdydz$$

E: solid under the cone $z = \sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2=1$, in the first octant.

STEP 1: Picture:



Date: Wednesday, November 3, 2021.

Note: $z = \sqrt{x^2+y^2} \Rightarrow \phi = \frac{\pi}{4}$ and $x^2+y^2+z^2 = 1 \Rightarrow \rho = 1$

STEP 2: Inequalities: (here we have $\frac{\pi}{2}$ because it's the first octant)

$$\begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \theta \leq \dfrac{\pi}{2} \\ \dfrac{\pi}{4} \leq \phi \leq \dfrac{\pi}{2} \end{cases}$$

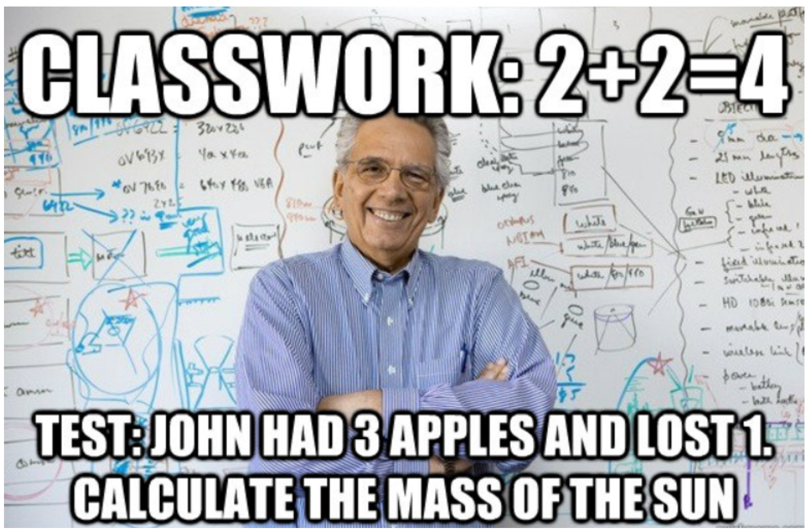
STEP 3: Integrate:

$$\begin{aligned}
&\int\int\int_E x \, dxdydz \\
=&\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho \sin(\phi)\cos(\theta)\, \rho^2 \sin(\phi)\, d\rho d\theta d\phi \\
=&\left(\int_0^1 \rho^3 d\rho\right)\left(\int_0^{\frac{\pi}{2}} \cos(\theta) d\theta\right)\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2(\phi) d\phi\right) \\
=&\left[\frac{\rho^4}{4}\right]_0^1 [\sin(\theta)]_0^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} - \frac{1}{2}\cos(2\phi) d\phi \\
=&\left(\frac{1}{4}\right)(1)\left[\frac{\phi}{2} - \frac{1}{4}\sin(2\phi)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \quad (u = 2\phi) \\
=&\frac{1}{4}\left(\frac{\pi}{4} - \frac{\pi}{8} - \frac{1}{4}\sin(\pi) + \frac{1}{4}\sin\left(\frac{\pi}{2}\right)\right) \\
=&\frac{1}{4}\left(\frac{\pi}{8} + \frac{1}{4}\right) \\
=&\frac{\pi}{32} + \frac{1}{16}
\end{aligned}$$

2. Mass of the sun

Video: Mass of the Sun

And of course I saved the best for last! Because there is this meme that was popular a couple of years ago:



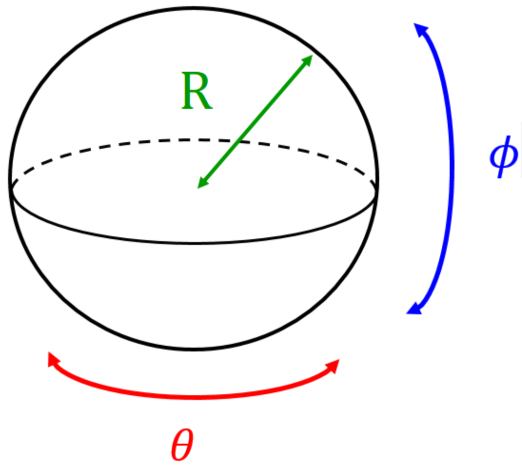
Without further ado . . . let's calculate the mass of the sun!

Example 2:

Suppose the sun E is a ball of radius $R = 6.9 \times 10^{10}$ cm and density $\frac{1}{\sqrt{x^2+y^2+z^2}}$ g/cm^3. What is the mass of the sun?

Note: $\frac{1}{\sqrt{x^2+y^2+z^2}}$ blows up near $(0, 0, 0)$, so this is saying that the sun is heavier at its core than on the surface, which makes sense physically.

STEP 1: Picture:



STEP 2: Inequalities:

$$\begin{cases} 0 \leq \rho \leq R \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{cases}$$

STEP 3: Integrate:

$$\begin{aligned} \text{Mass } &= \int\int\int_E \frac{1}{\sqrt{x^2+y^2+z^2}}\, dxdydz \\ &= \int_0^{\pi}\int_0^{2\pi}\int_0^R \left(\frac{1}{\rho}\right) \rho^2 \sin(\phi) d\rho d\theta d\phi \\ &= 2\pi \left(\int_0^R \rho d\rho\right)\left(\int_0^{\pi} \sin(\phi) d\phi\right) \\ &= (2\pi)\frac{R^2}{2}(2) \\ &= 2\pi R^2 \\ &\approx 2.99 \times 10^{22} g \end{aligned}$$

Remark: NASA uses the following density:

$$519\left(\frac{\rho}{R}\right)^4 - 1630\left(\frac{\rho}{R}\right)^3 + 1844\left(\frac{\rho}{R}\right)^2 - 889\left(\frac{\rho}{R}\right) + 155$$

Which would give you 2.7×10^{33} grams. The actual mass is 1.98×10^{33} grams, so this is not bad at all!

3. $u-$sub the COOL way

Welcome to the one and only integration technique in this course: $u-$sub! For this, let me "remind" you how to do single-variable $u-$sub, but I'll present it in a way that will be useful in this course

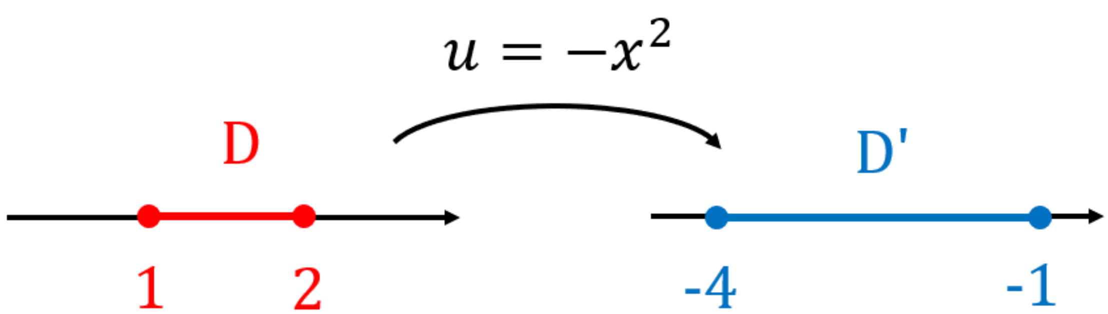
Example 3:

Calculate $\int_1^2 e^{-x^2}(-2x)dx$

STEP 1: Let $u = -x^2$

STEP 2: **Endpoints:** $u(1) = -1, u(2) = -4$.

So u turns $D = [1, 2]$ into $D' =$ ~~$[-1, -4]$~~ $= [-4, -1]$.



STEP 3: du: Beware of the absolute value! (makes sense, du should be positive)

$$du = \left|\frac{du}{dx}\right| dx = |-2x| dx = 2x dx \Rightarrow -2x dx = -du$$

STEP 4: Integrate

$$\int_1^2 e^{-x^2}(-2x)dx = \int_{[1,2]} e^{-x^2}(-2x)dx = \int_D e^{-x^2}(-2x)dx = \int_{D'} e^u(-du)$$
$$= -\int_{[-4,-1]} e^u du = -\int_{-4}^{-1} e^u du = -\left(e^{-1} - e^{-4}\right) = e^{-4} - e^{-1}$$

4. Multivariable Example

Video: The Jacobian

The good news is that for double and triple integrals, the process is similar to the above!

Example 4:

$$\int\int_D \sin\left(\frac{y-x}{y+x}\right) dxdy$$

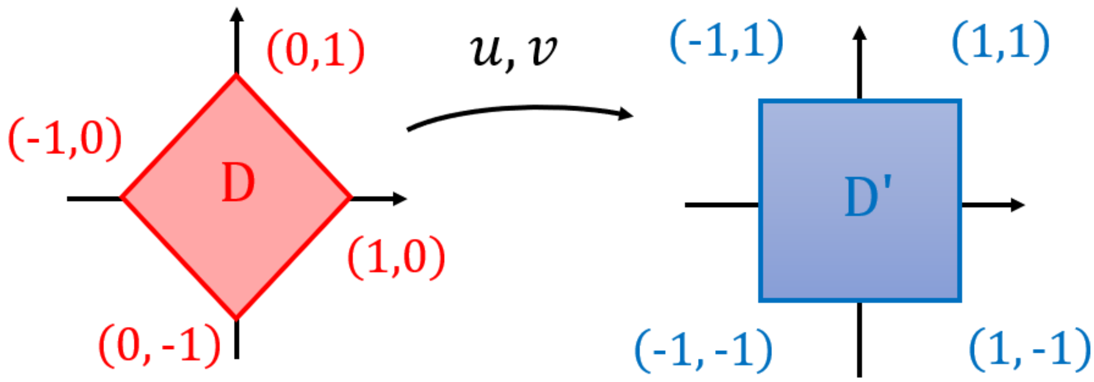
Where D is the square with vertices $(-1,0), (0,-1), (1,0), (0,1)$.

STEP 1:

$$\begin{cases} u = y - x \\ v = y + x \end{cases}$$

STEP 2: "Endpoints"

Trick: Look at the values of u and v at the vertices:



$$(-1,0) \Rightarrow \begin{cases} u = y - x = 0 - (-1) = 1 \\ v = y + x = 0 + (-1) = -1 \end{cases} \Rightarrow (1,-1)$$

$$(0,-1) \Rightarrow \begin{cases} u = -1 - 0 = -1 \\ v = -1 + 0 = -1 \end{cases} \Rightarrow (-1,-1)$$

Similarly $(1,0)$ becomes $(-1,1)$ and $(0,1)$ becomes $(1,1)$.

So D' is a square with vertices $(1,-1), (-1,-1), (-1,1), (1,1)$

STEP 3: "$du = \left|\frac{du}{dx}\right| dx$"

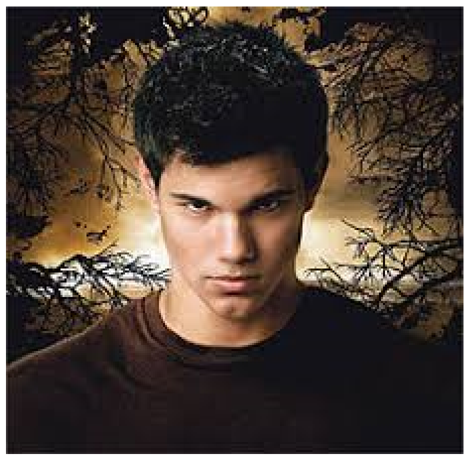
$$\text{Here we get: } dudv = \left|\frac{dudv}{dxdy}\right| dxdy$$

Let's put all the possible partial derivatives together in a determinant:

$$\frac{dudv}{dxdy} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = (-1)(1) - (1)(1) = -2$$

$$\text{Therefore: } dudv = |-2|dxdy = 2dxdy \Rightarrow dxdy = \frac{1}{2}dudv$$

Note: This number (or its absolute value) is called the Jacobian, a tribute to Taylor Lautner in Twilight. It's sometimes written as $\frac{\partial(u,v)}{\partial(x,y)}$ instead of $\frac{dudv}{dxdy}$



STEP 4: Integrate:

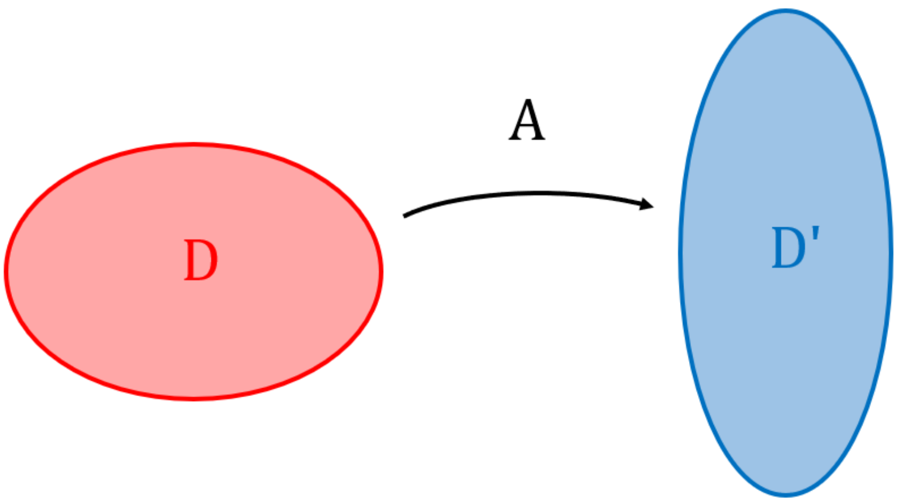
$$\begin{aligned}\int\int_D \sin\left(\frac{y-x}{y+x}\right) dxdy &= \int\int_{D'} \sin\left(\frac{u}{v}\right)\frac{1}{2}dudv \\ &= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\sin\left(\frac{u}{v}\right)dudv \quad \text{(Much easier to integrate)} \\ &= \frac{1}{2}\int_{-1}^{1} 0\, dv \quad \text{(Because } \sin\left(\frac{u}{v}\right) \text{ is odd in } u\text{)} \\ &= 0\end{aligned}$$

5. Optional Appendix: Why this works

Fact from Linear Algebra:

If D and D' are regions and A is a matrix between them, then:

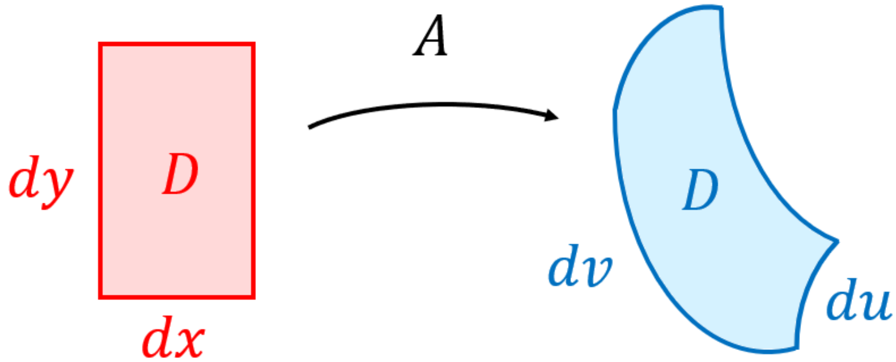
$$\text{Area}(D') = |\det(A)| \, \text{Area}(D)$$



Suppose that D is a small rectangle with sides dx and dy. Then

$$A = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

transforms D into D', which is an object with sides du and dv:



On the one hand, the area of D' is approximately $dudv$, but on the other hand, by the formula above:

$$\text{Area}(D') = |\det A| \, \text{Area}(D)$$

$$dudv = \left| \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \right| dxdy$$

$$dudv = \left| \frac{dudv}{dxdy} \right| dxdy$$

Finally, multiply both sides of the above by $f(u,v) = f(x,y)$ and integrate to get:

$$\int\int_{D'} f(u,v) dudv = \int\int_{D} f(x,y) \left| \frac{dudv}{dxdy} \right| dxdy$$