LECTURE 3: THE COMPLETENESS AXIOM (I)

Today: We'll discuss the *single* most important concept in this course: The Supremum of a set. It's basically a more relaxed version of a maximum, so let's cover that first.

1. Maximum and Minimum

Video: Max and Min

Scenario: Suppose I tell you: "You're the best student in the class." This means that you're better than all the other students in the class. Same with maximum:

Definition:

Let S be a nonempty subset of $\mathbb{R}$.

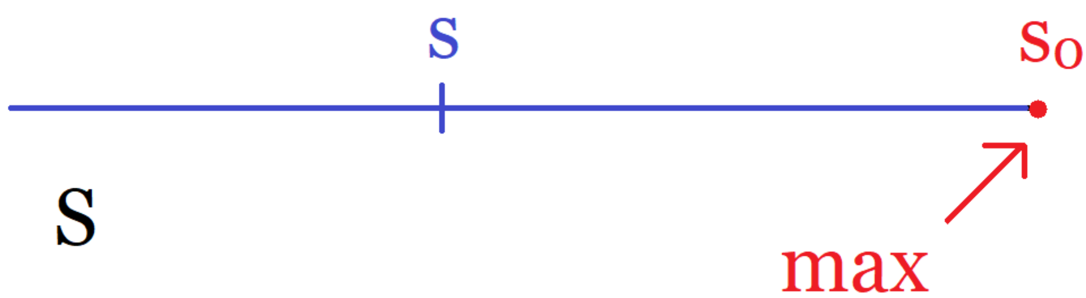
We say s_0 is the **maximum** of S and write $s_0 = \max S$ if

(1) $s_0 \in S$

(2) For all $s \in S$, $s \leq s_0$

Similar for $\min S$ (but this time $s \geq s_0$)

Date: Tuesday, September 7, 2021.



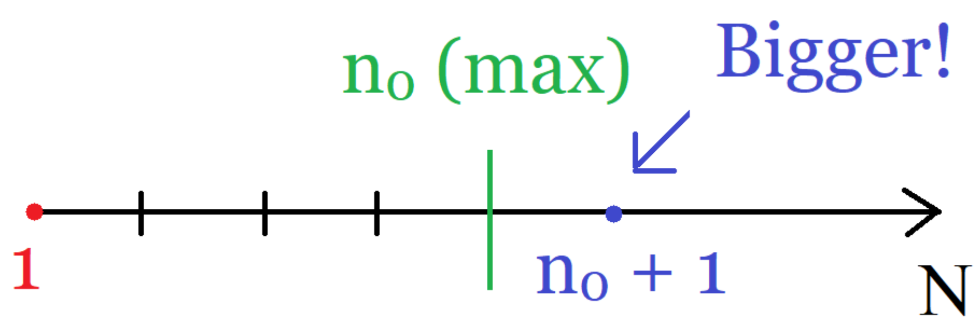
Examples:

(1) $S = \{1, 2, 3, 4\}$, then $\max S = 4$

(2) $S = [0, 1]$, then $\max S = 1$ and $\min S = 0$

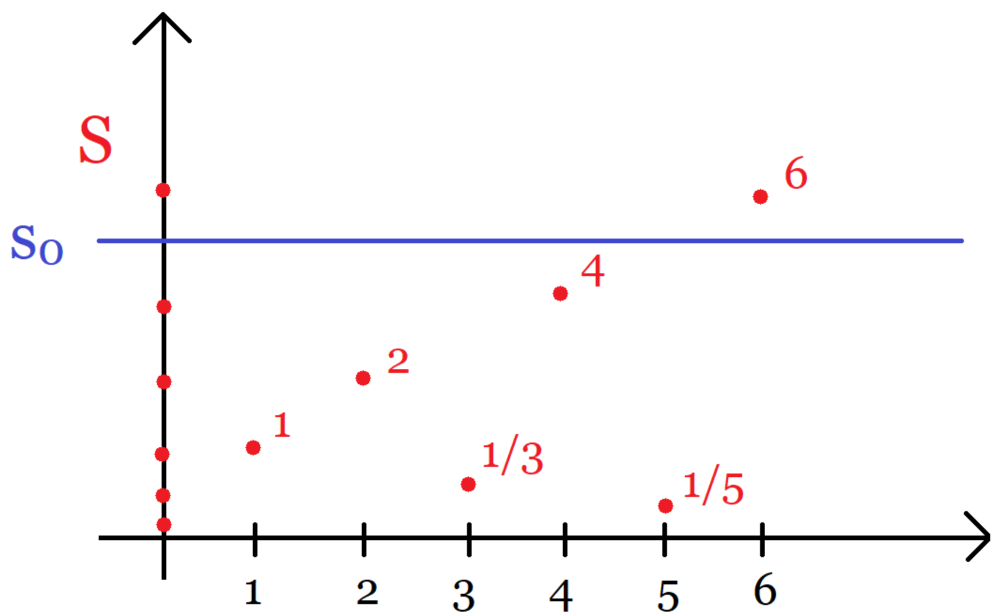
(3) $\min \mathbb{N} = 1$, but $\mathbb{N}$ has no max:

Why? Suppose $\max \mathbb{N} = n_0$, then by definition $n \leq n_0$ for all $n \in \mathbb{N}$. But let $n = n_0 + 1$ to get $\underbrace{n_0 + 1}_{n} \leq n_0 \Rightarrow\Leftarrow$



(4) (extra practice)

$$S = \left\{ n^{(-1)^n} \mid n \in \mathbb{N} \right\} = \{1^{-1}, 2^1, 3^{-1}, 4^1, \cdots\} = \left\{ 1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \cdots \right\}$$



Then S has no max: Suppose $\max S = s_0$, then by definition for all $n \in \mathbb{N}$, $n^{(-1)^n} \leq s_0$ for all n.

But then let n be any **even** integer that is greater than s_0, then $n^{(-1)^n} = n > s_0$ which contradicts $n^{(-1)^n} \leq s_0 \Rightarrow\Leftarrow$

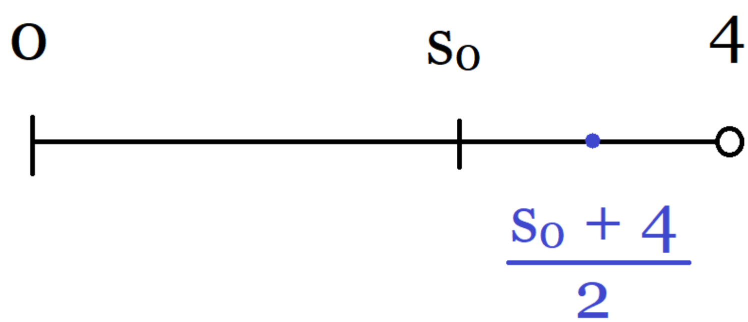
(Also S has no min, but we'll come back to that later)

Important Note:

It's important that $\max S$ be an element S (it's part of the *definition* of max).

(5) $S = [0, 4)$. Then $\min S = 0$, but $\max S$ doesn't exist (it *cannot* be 4 because $4 \notin S$).

Why? Suppose $\max S = s_0$. Then for all $s \in S$, $s \leq s_0$. But then let $s = \frac{s_0+4}{2} \in S$ (midpoint of s_0 and 4), then by construction $s > s_0$, but this contradicts $s \leq s_0 \Rightarrow\Leftarrow$



This is really bothersome though! *Intuitively* 4 should be the max of S, but the *only* reason it isn't is because here 4 is not in S

Question:

Is there a way to relax the notion of max so that $\max[0, 4) = 4$? (even though 4 is not in it)

It turns out there is, and it's called the sup!

2. Upper and Lower Bounds

Video: Upper Bound

The notion of sup has to do upper bounds, which we'll define now:

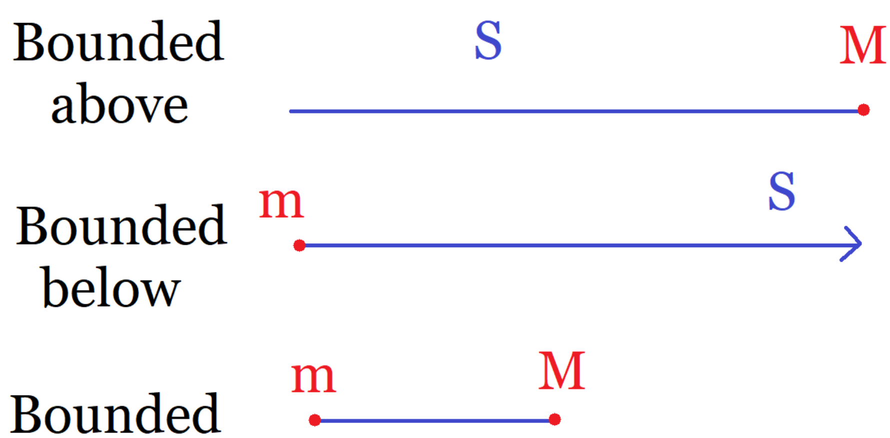
Definition:

Let S be a nonempty subset of $\mathbb{R}$. Then:

(1) S is **bounded above** by $M < \infty$ if $s \leq M$ for all $s \in S$. We call M an **upper bound** for S

(2) S is **bounded below** by $m > -\infty$ is $s \geq m$ for all $s \in S$. We call m a **lower bound** for S

(3) Finally, S is **bounded** if it is both bounded above and below.



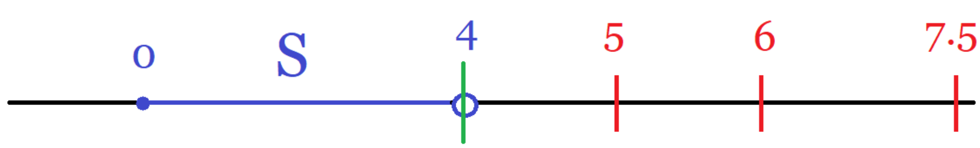
Examples:

(1) $\mathbb{N}$ is bounded below by 1 but not bounded above, so $\mathbb{N}$ is unbounded.

(2) $S = [0, 4)$ is bounded above by 4 since $s \leq 4$ for all $s \in S$. It is also bounded below by 0 since $s \geq 0$ for all $s \in S$. Hence S is bounded

Remark: The upper bound M doesn't have to be in S; that's what makes this so great!

Important Observation: 4 is an upper bound for S, but there are many other upper bounds for S, like $5, 6, 7.5$, and in fact any number ≥ 4 is an upper bound for S.



Intuitively it seems that 4 is the "optimal" upper bound. And in fact, among all the possible upper bounds of S, 4 is the smallest one. This finally brings us to the concept of sup, also called the *least* upper bound:

3. Supremum

Video: What's Sup?

Intuitively: $\sup(S)$ is the smallest possible upper bound for S (like 4 in the previous example).

The book in fact takes that to be the definition of sup, but the definition below is more widely used in analysis.

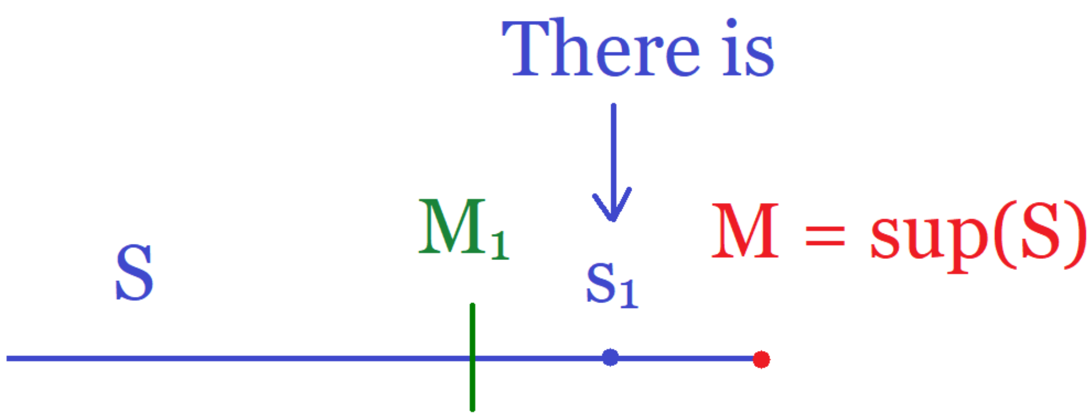
Analogy: Suppose I tell you: "You did not get the highest grade in the course?" Your first reaction should be: "Who got a *higher* grade than me?" And this is the point of view that we'll take.

Definition:

Let M be an upper bound for S.

Then $M = \sup(S)$ (M is the **supremum** of S) if:

$$\text{For all } M_1 < M \text{ there is } s_1 \in S \text{ such that } s_1 > M_1$$



In terms of the analogy: If you (M_1) didn't get the best grade (M), then there is someone (s_1) who got a better grade than you ($s_1 > M_1$)

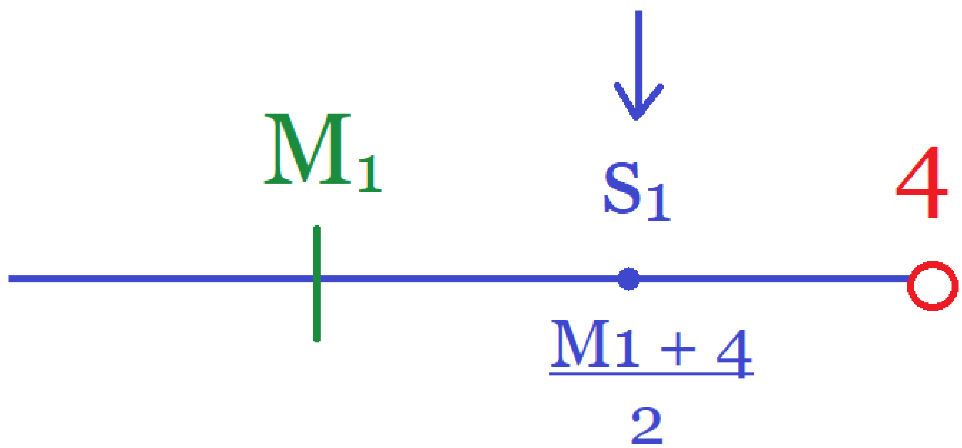
What this is saying is that M is really the *least* upper bound. Any M_1 smaller than M *cannot* be an upper bound (since there is an element

s_1 in S bigger than it)

Examples:

(1) $S = (-\infty, 4)$, then $\sup(S) = 4$.

Why? First of all, 4 is an upper bound for S. Now suppose $M_1 < 4$ then we want to find (WTF) $s_1 \in S$ such that $s_1 > M_1$.

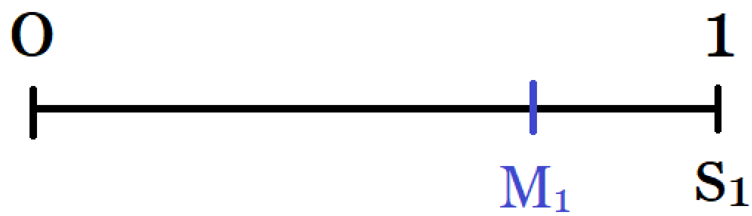


Let $s_1 = \frac{M_1+4}{2}$ (midpoint between M_1 and 4) then $s_1 \in S$ (since $s_1 < 4$) but also $s_1 > M_1$ (by construction) ✓

Note: So in this sense 4 is the "maximum" of S, even though it's technically not in S

(2) $S = [0, 1]$, then $\sup(S) = 1$.

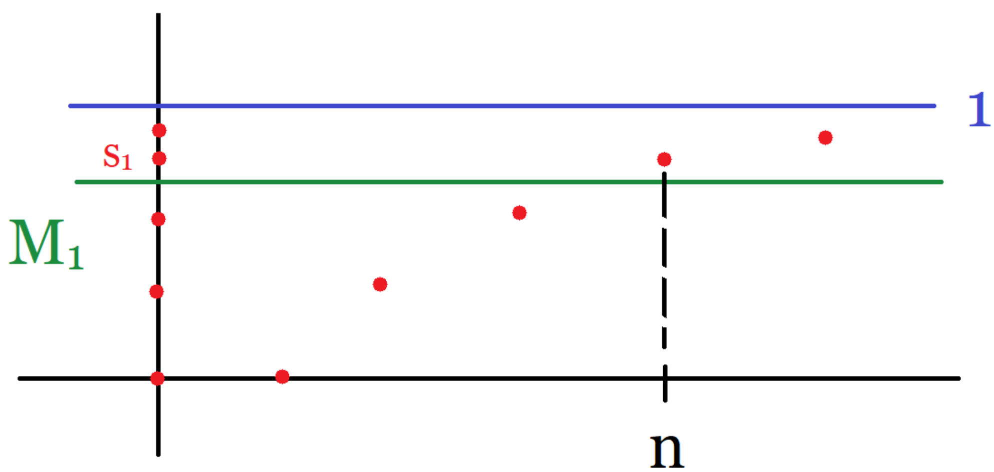
Why? 1 is an upper bound, and if $M_1 < 1$, let $s_1 = 1 \in S$, then $s_1 = 1 > M_1$ (since $M_1 < 1$) ✓



In fact, if $\max(S)$ exists, then $\sup(S) = \max(S)$, but the point is that $\sup(S)$ is more general than a max.

(3) (extra practice, useful for the HW)

$$S = \left\{ 1 - \frac{1}{n}, n \in \mathbb{N} \right\}$$



Claim: $\sup(S) = 1$

Why? 1 is an upper bound since $1 - \frac{1}{n} \leq 1$ for all n. Now let $M_1 < 1$, need to find $s_1 \in S$ such that $s_1 > M_1$. Then notice:

$$\begin{aligned} 1-\frac{1}{n} &> M_1 \\ \Leftrightarrow \frac{1}{n} &< 1-M_1 \\ \Leftrightarrow n &> \frac{1}{1-M_1} \end{aligned}$$

So let n be any integer greater than $\frac{1}{1-M_1} > 0$ and let $s_1 = 1-\frac{1}{n}$, then by the above we get $s_1 = 1-\frac{1}{n} > M_1$ ✓

4. The Least Upper Bound Property

Video: Least Upper Bound Property

Note: What makes sup so special? Remember that $\max(S)$ doesn't always exist. For instance, in the case $S = [0, 4)$, $\max(S)$ doesn't exist.

But this is not the case with sup. The following theorem, which is really the **fundamental theorem of analysis**, says that $\sup(S)$ *always* exists:[1]

Least Upper Bound Property

If S is a nonempty subset of $\mathbb{R}$ that is bounded above, then S has a least upper bound, that is $\sup(S)$ exists.

Think of this theorem as saying "$\sup(S)$ always exists." Because either S is bounded above (in which case $\sup(S)$ exists), or S is unbounded

[1] The book calls it the Completeness Axiom

(in which case $\sup(S) = \infty$)

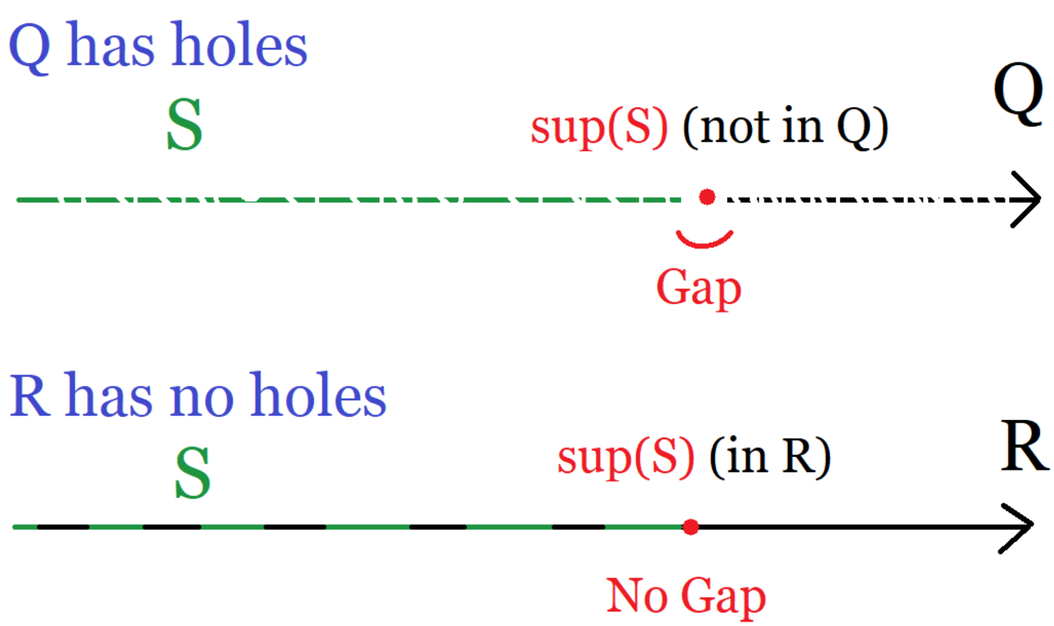
Note: Geometrically, this theorem is saying that $\mathbb{R}$ is complete, that is it does not have any gaps/holes.

Non-Example: The property is **NOT** true for $\mathbb{Q}$. Let:

$$S = \left\{ x \in \mathbb{Q} \mid x^2 < 2 \right\}$$

Then S is bounded above[2] by 3., but it doesn't have a least upper bound in $\mathbb{Q}$ because $\sup(S) = \sqrt{2}$ but $\sqrt{2}$ isn't in $\mathbb{Q}$.

In some sense, $\mathbb{Q}$ is broken: It has holes and gaps where the sup is supposed to be!



[2]Because if $x > 3$, then $x^2 > 9 \geq 2$, so x cannot be in S, so by the contrapositive $x \in S \Rightarrow x \leq 3$

$\mathbb{R}$ doesn't have that problem, it is **complete**, it has no holes, the sup is exactly where it's supposed to be!

Fun Fact: It's always possible to fix a broken heart; it's always possible to complete a space with holes.

5. Infimum

Video: Infimum

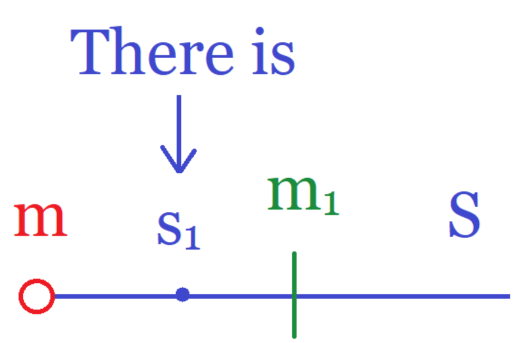
On the other side of the coin, there is the concept of Infimum, which is a generalization of minimum.

Analogy: If you didn't get the lowest grade, then someone got a lower grade than you.

Definition:

Let m be a lower bound for S. Then $m = \inf(S)$ (m is the **infimum** of S) if:

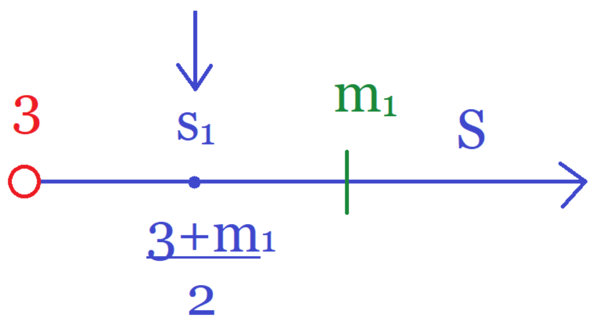
$$\text{For all } m_1 > m \text{ there is } s_1 \in S \text{ such that } s_1 < m_1$$



Examples:

(1) $S = (3, \infty)$, then $\inf(S) = 3$

Why? First, 3 is a lower bound. Now suppose $m_1 > 3$, need to find $s_1 \in (3, \infty)$ such that $s_1 < m_1$. Again, $s_1 = \frac{3+m_1}{2}$ (midpoint between 3 and m_1) does the trick ✓



(2) If $S = [0, 1]$, $\inf(S) = 0$ (similar to before) In fact if $\min(S)$ exists, then $\inf(S) = \min(S)$, so inf is a generalization of min

(3) (extra practice)

$$S = \left\{ n^{(-1)^n}, n \in \mathbb{N} \right\} = \left\{ 1, 2, \frac{1}{3}, 4, \frac{1}{5}, \dots \right\}$$

We claim that $\inf(S) = 0$. First, 0 is a lower bound since $n^{(-1)^n} \geq 0$ for all n. Now suppose $m_1 > 0$ $(= m)$, need to show that there is $s_1 \in S$ such that $s_1 < m_1$. That is, need to find n such that $n^{(-1)^n} < m_1$.

Notice that even powers of n just give bigger and bigger numbers like $1, 2, 4$ but odd powers give smaller and smaller numbers

Suppose n is odd, then $n^{(-1)^n} = n^{-1} = \frac{1}{n}$, and moreover $\frac{1}{n} < m_1 \Leftrightarrow n > \frac{1}{m_1}$

Hence let n be any **odd** integer $> \frac{1}{m_1}$, then if $n^{(-1)^n} \in S$ but $n^{(-1)^n} > m_1$ ✓

6. Inf vs Sup

Video: $\inf(S) = -\sup(-S)$

Are inf and sup related? Yes, in a really elegant way!

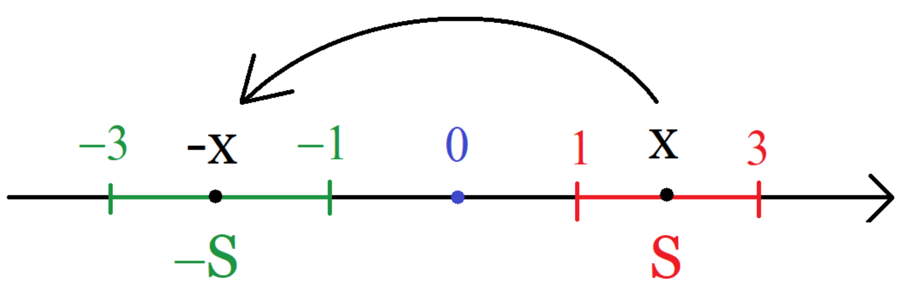
Definition:

If S is any subset of $\mathbb{R}$,then

$$-S = \{-x \mid x \in S\}$$

(In other words, reflect S across the origin)

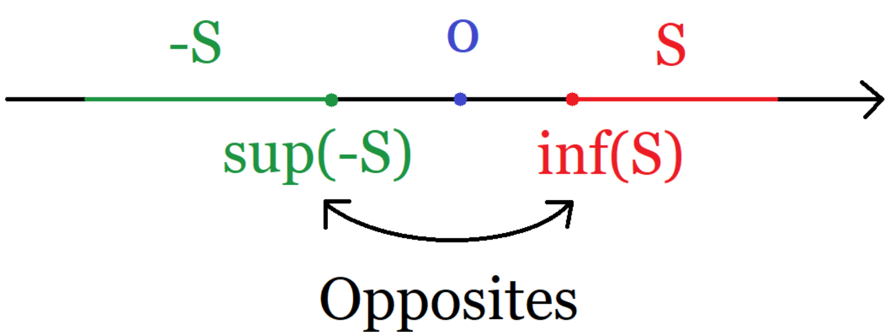
Example: If $S = [1,3]$ then $-S = [-3,-1]$



Notice: In this example, $\inf(S) = 1$, but also $\sup(-S) = -1$, so $\inf(S) = -\sup(-S)$, and in fact this is always true:

Important Fact (memorize!)

$$\inf(S) = -\sup(-S)$$



What this is saying is that anything that is true about sup is also true for inf. We'll see a consequence below.

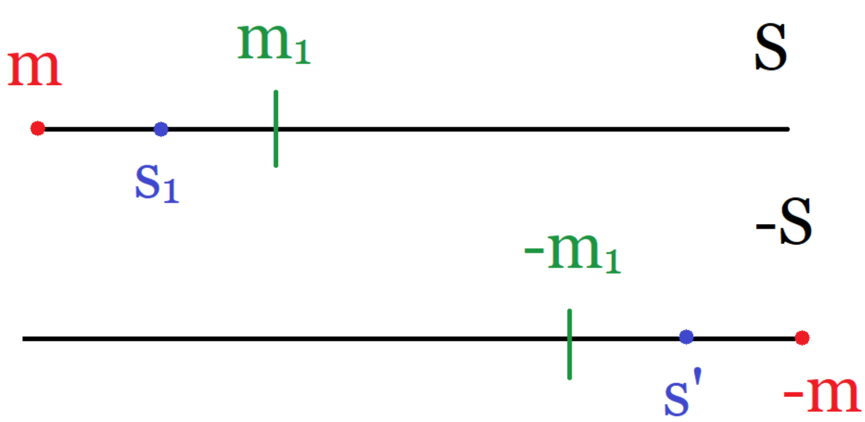
Proof of Important Fact:

Let $\boxed{m = -\sup(-S)}$

Then $\inf(S) = -\sup(-S) \Leftrightarrow \inf(S) = m$.

In order to show $\inf(S) = m$, we need to show S is bounded below by m (skip[3]) and: If $m_1 > m$ then there is $s_1 \in S$ such that $s_1 < m_1$.

[3]Since $\sup(-S) = -m$, $-S$ is bounded above by $-m$, so for all $(-s) \in -S$, $-s \leq -m \Rightarrow s \geq m$ for all $s \in S$



Suppose $m_1 > m$. Then $-m_1 < -m = sup(-S)$, so by definition of sup, there is $s' \in -S$ such that $s' > -m_1$.

But by definition of $-S$, $s' = -s_1$ for some $s_1 \in S$.

Then this s_1 works because $s' > -m_1 \Rightarrow -s_1 > -m_1 \Rightarrow s_1 < m_1$, which is what we wanted to show $\square$

Why useful? This basically says that you never have to prove statements with inf: Just prove the version with sup and use this theorem. In fact, let's illustrate this with:

Greatest Lowest Bound Property

If S is a nonempty subset of $\mathbb{R}$ that is bounded below, then $\inf(S)$ exists.

Proof: Suppose S is a nonempty subset that is bounded below by m, then for all $s \in S$, $s \geq m > -\infty$, so for all $s \in S$, $-s \leq -m$. This says that $-S$ is bounded above by $-m < \infty$.

By the Least Upper Bound Property, $\sup(-S)$ exists, and therefore $\inf(S)$ exists because

$$\inf(S) = -\underbrace{\sup(-S)}_{\text{Exists}} \quad \square$$