LECTURE 3: THE CROSS PRODUCT

Today: We learn about a cool new way of multiplying vectors, called the **cross product**. This uses the concept of a determinant, which I define first.

1. Determinants

Video: Determinants and Bomberman

Video: Area of a Parallelogram

Definition: (2×2 determinant)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

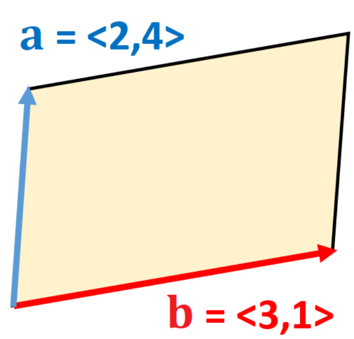
Example 1:

$$\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} = (5)(4) - (2)(3) = 14$$

Determinants arise naturally in calculating areas of parallelograms

Example 2:

Find the area of the parallelogram with sides $\mathbf{a} = \langle 2, 4 \rangle$ and $\mathbf{b} = \langle 3, 1 \rangle$



Date: Friday, January 3, 2021.

Calculate the determinant with rows $\langle 2, 4\rangle$ and $\langle 3, 1\rangle$:

$$\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = (2)(1) - (3)(4) = -10$$

Therefore the area is $|-10| = 10$

Don't forget about the absolute value (which makes sense, since areas are positive)

For 3×3 determinants, the formula is slightly more complicated:

Example 3: (3×3 Determinants)

Calculate the following determinant:

$$\begin{vmatrix} 2 & 0 & 3 \\ 1 & 2 & 1 \\ 4 & 1 & 2 \end{vmatrix}$$

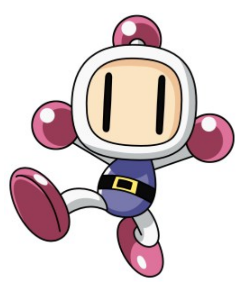
STEP 1: Here we will expand the determinant along the first row, meaning that the answer will be of the form:

$$\begin{vmatrix} 2 & 0 & 3 \\ 1 & 2 & 1 \\ 4 & 1 & 2 \end{vmatrix} = +2 \times \begin{vmatrix} ? & ? \\ ? & ? \end{vmatrix} - 0 \times \begin{vmatrix} ? & ? \\ ? & ? \end{vmatrix} + 3 \times \begin{vmatrix} ? & ? \\ ? & ? \end{vmatrix}$$

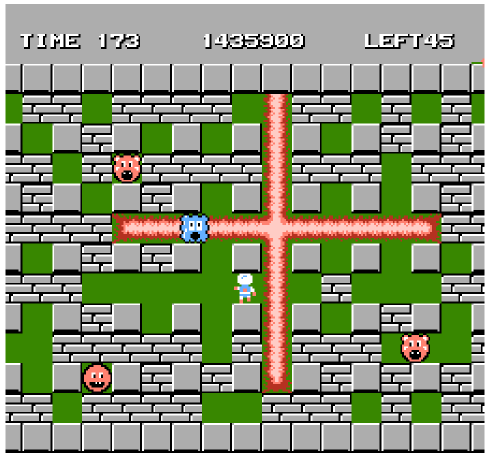
(Here the signs are always $+, -, +$ and the numbers $2, 0, 3$ are the entries from the first row)

STEP 2: How to fill out the ?

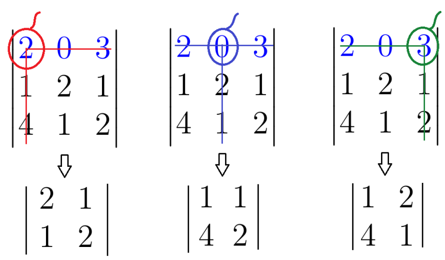
It's Bomberman time!!!



Analogy: In the video game *Super Bomberman*, the main character places a bomb, which destroys everything in the same row and column as the bomb, as follows:



It's the same thing with determinants: You first place a bomb at 2 (the first entry) and take the determinant of the rest, that is $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$, then you place a bomb at 0 to get $\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix}$ and finally you place a bomb at 3 to get $\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$



Therefore we get:

$$\begin{aligned} \begin{vmatrix} 2 & 0 & 3 \\ 1 & 2 & 1 \\ 4 & 1 & 2 \end{vmatrix} &= +2 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 0 \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} \\ &= 2(4-1) - 0(2-4) + 3(1-8) \\ &= 6 - 21 \\ &= -15 \end{aligned}$$

Example 4: (extra practice)

Calculate the following determinant:

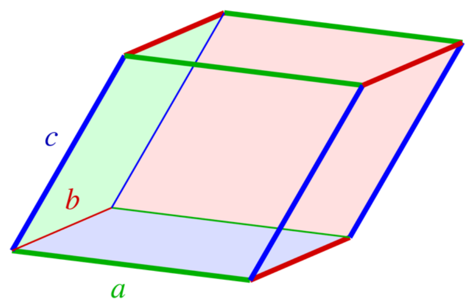
$$\begin{vmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{vmatrix} = +3 \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} - 6 \begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} + 7 \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix}$$
$$= 3(8-3) - 6(-2) + 7(-4)$$
$$= 15 + 12 - 28$$
$$= -1$$

Appliction: Just like 2×2 determinants are used to calculate areas of parallelograms, 3×3 determinants are used to calculate volumes of parallelipipeds.

Example 5: (Application)

Find the volume of the parallelipiped with sides $\mathbf{a} = \langle 1, 2, 3 \rangle$, $\mathbf{b} = \langle 1, -1, -2 \rangle$, $\mathbf{c} = \langle 2, 1, 4 \rangle$



First calculate the determinant whose rows are $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & -2 \\ 2 & 1 & 4 \end{vmatrix} = 1 \begin{vmatrix} -1 & -2 \\ 1 & 4 \end{vmatrix} - 2 \begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix} + 3 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$
$$= (-4+2) - 2(4+4) + 3(1+2)$$
$$= -2 - 16 + 9$$
$$= -9$$

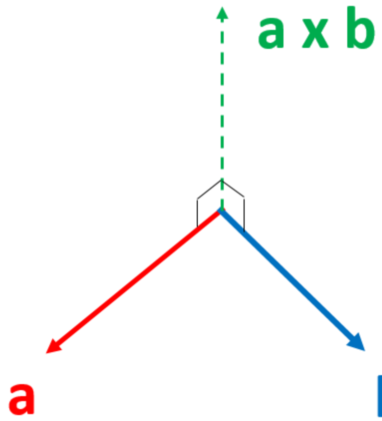
Then the volume is $|-9| = 9$

Note: The book uses the formula $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ (where $\times$ is the cross product), but this one is a bit easier to use.

2. Cross Products

Using the determinant, we can finally define the concept of a cross product.

Intuitively: Given two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$



Example 6:

Calculate $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle 4, 5, 6 \rangle$

The cross product is just a special determinant

(recall that $\mathbf{i} = \langle 1, 0, 0 \rangle, \mathbf{j} = \langle 0, 1, 0 \rangle, \mathbf{k} = \langle 0, 0, 1 \rangle$)

Definition:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

Which here becomes:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} =& \mathbf{i} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \\ =& \mathbf{i}(12 - 15) - \mathbf{j}(6 - 12) + \mathbf{k}(5 - 8) \\ =& -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k} \\ =& \langle -3, 6, -3 \rangle \end{aligned}$$

Important: $\mathbf{a} \times \mathbf{b}$ is a **vector**, not a number. It is a vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. In fact you can check that $\langle 3, 6, -3 \rangle$ is perpendicular to both $\langle 1, 2, 3 \rangle$ and $\langle 4, 5, 6 \rangle$ (the dot product is 0)

Example 7: (extra practice)

Calculate $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = \langle 2, -1, 3 \rangle$ and $\mathbf{b} = \langle 4, 2, 1 \rangle$

$$\begin{aligned}
\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 4 & 2 & 1 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} \\
&= \mathbf{i}(-1-6) - \mathbf{j}(2-12) + \mathbf{k}(4+4) \\
&= -7\mathbf{i} + 10\mathbf{j} + 8\mathbf{k} \\
&= \langle -7, 10, 8 \rangle
\end{aligned}$$

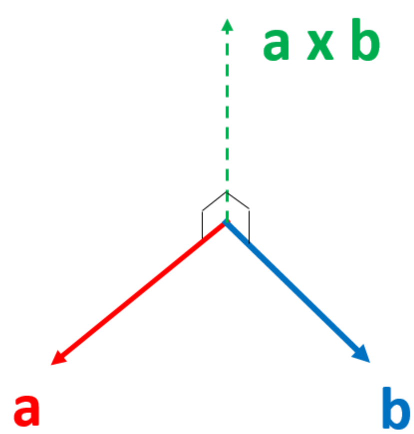
3. Properties

Why is the cross product so useful?

Single Most Important Property

$\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$

In particular, this allows us to find a vector that is perpendicular to two other ones. This will be **SUPER** useful when we'll talk about planes (in a couple of lectures).

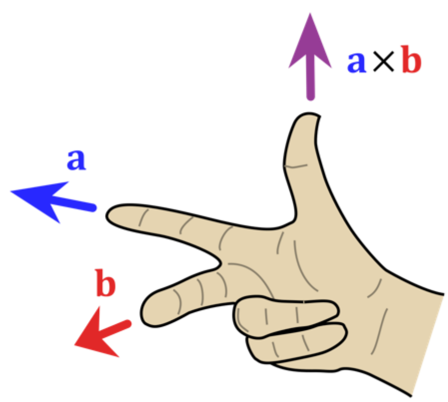


Example 8:

Find a vector perpendicular to both $\mathbf{a} = \langle 0, 1, 7 \rangle$ and $\mathbf{b} = \langle 2, -1, 4 \rangle$

$$
\begin{aligned}
\mathbf{a} \times \mathbf{b} = & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 7 \\ 2 & -1 & 4 \end{vmatrix} \\
= & \mathbf{i} \begin{vmatrix} 1 & 7 \\ -1 & 4 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 0 & 7 \\ 2 & 4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} \\
= & \mathbf{i}(4+7) - \mathbf{j}(-14) + \mathbf{k}(-2) \\
= & 11\mathbf{i} + 14\mathbf{j} - 2\mathbf{k} \\
= & \langle 11, 14, -2 \rangle
\end{aligned}
$$

Note: The direction of $\mathbf{a} \times \mathbf{b}$ is determined by the right-hand-rule: If you curl your hand from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.



Other Properties (not as important)

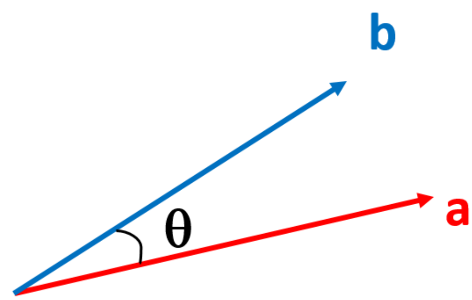
(1) **Parallel:** (here $\mathbf{0} = \langle 0, 0, 0 \rangle$)

$$\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a} \text{ and } \mathbf{b} \text{ are } \textbf{parallel}$$

(2) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ (since $\mathbf{a}$ is parallel to itself)
(3) $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$
(4) $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
(5) **Angle Formula:**

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\theta)$$

(Where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$)

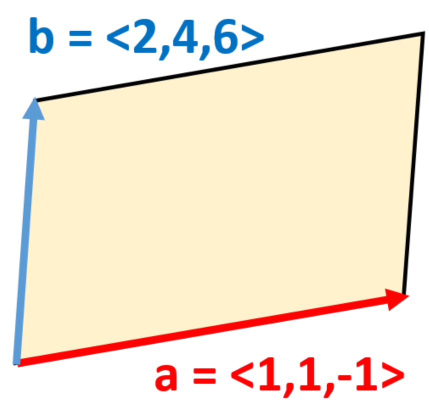


4. Areas of a parallelogram (again)

Surprisingly, we can also use cross products to calculate areas of parallelograms, but this time in 3 dimensions:

Example 9:

Find the area of the paralellogram with sides $\mathbf{a} = \langle 1, 1, -1 \rangle$ and $\mathbf{b} = \langle 2, 4, 6 \rangle$



$$\begin{aligned}
\mathbf{a} \times \mathbf{b} = & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{vmatrix} \\
= & \mathbf{i} \begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} \\
= & \mathbf{i}(6+4) - \mathbf{j}(6+2) + \mathbf{k}(4-2) \\
= & 10\mathbf{i} - 8\mathbf{j} + 2\mathbf{k} \\
= & \langle 10, -8, 2 \rangle
\end{aligned}$$

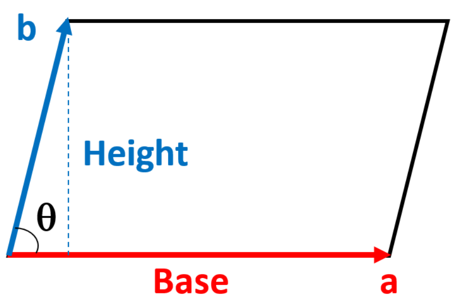
Fact:

Area of parallelogram $= \|\mathbf{a} \times \mathbf{b}\|$

So here the answer is

$$\sqrt{10^2 + (-8)^2 + 2^2} = \sqrt{100 + 64 + 4} = \sqrt{168}$$

Why? (will probably skip) The reason for this is surprisingly elegant!



Consider the parallelogram with sides **a** and **b**.

The area of the parallelogram is given by:

$$\text{Area} = \text{Base} \times \text{Height}$$

From the picture above, we have Base $= \|\mathbf{a}\|$

Moreover, from SOHCAHTOA, we have

$$\sin(\theta) = \frac{\text{Height}}{\|\mathbf{b}\|} \Rightarrow \text{Height} \ = \|\mathbf{b}\| \sin(\theta)$$

Therefore the area of the parallelogram becomes

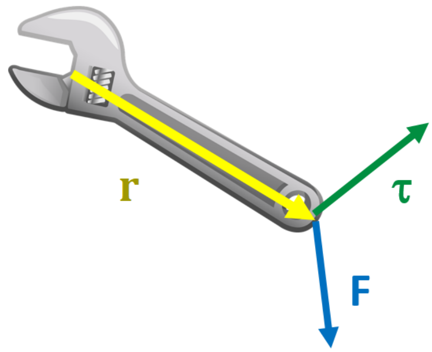
$$\text{Base} \ \times \ \text{Height} \ = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\theta) = \|\mathbf{a} \times \mathbf{b}\|$$

Where in the last step we used the Angle formula for cross-products □

5. Torque

Finally, cross products appear naturally in physics as the torque of an object.

Setting: Suppose you're applying a force **F** to a wrench at a position **r**



Definition:

The **torque** τ is $\tau = \mathbf{F} \times \mathbf{r}$

It measures the tendency of the body to rotate about the origin.

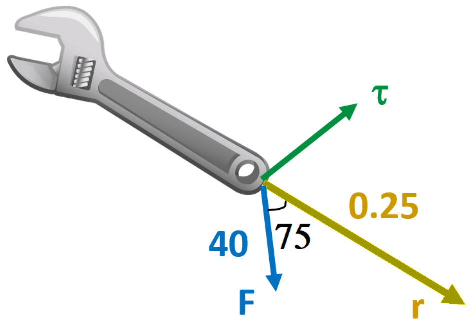
Example 10: (extra practice)

Find the torque obtained when applying a force $\mathbf{F} = \langle 1, 2, 1 \rangle$ to a position $\mathbf{r} = \langle 3, 1, 5 \rangle$

$$\begin{aligned}
\tau =& \mathbf{F} \times \mathbf{r} \\
=& \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 3 & 1 & 5 \end{vmatrix} \\
=& \mathbf{i} \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} \mathbf{j} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \\
=& (10 - 1)\mathbf{i} - (5 - 3)\mathbf{j} + (1 - 6)\mathbf{k} \\
=& 9\mathbf{i} - 2\mathbf{j} - 5\mathbf{k} \\
=& \langle 9, -2, -5 \rangle
\end{aligned}$$

Example 11: (extra practice)

Suppose you apply a force of 40 N to a wrench that is 0.25 m long, where the angle between the force and the position is 75 degrees. Find the magnitude of the torque.



By the Angle Formula:

$$\begin{aligned}\|\tau\| &= \|\mathbf{F} \times \mathbf{r}\| \\ &= \|\mathbf{F}\| \, \|\mathbf{r}\| \sin(\theta) \\ &= (40)(0.25)\sin(75^\circ) \\ &= 10\sin(75^\circ) \\ &\approx 9.66 N \times m \\ &= 9.66 J\end{aligned}$$