LECTURE 3: EQUICONTINUITY

1. Introduction

One of the cornerstone theorems in Analysis 1 is the celebrated Bolzano-Weierstraß Theorem, which says:

Bolzano-Weierstraß: If (s_n) is a bounded sequence of real numbers, then (s_n) has a convergent subsequence (s_{n_k})

For example, it is used to prove the Extreme Value Theorem

Question: Is B-W still true for functions? That is: if (f_n) is a bounded sequence of functions, does it have a uniformly convergent subsequence (f_{n_k})?

Unfortunately the answer is no ☹

Definition: A sequence (f_n) on $[a, b]$ is (uniformly) **bounded** if there is M such that for all n and all x we have

$$|f_n(x)| \leq M$$

Note: Rudin distinguishes bounded and uniformly bounded, but on the compact interval $[a, b]$, they're the same thing.

Date: Thursday, July 6, 2022.

2. Counterexample

Non-Example: Consider the sequence $f_n(x) = \sin(nx)$ on $[0, 2\pi]$

Then $|f_n(x)| = |\sin(nx)| \leq 1$, so f_n is bounded.

Suppose f_n had a uniformly convergent subsequence $f_{n_k} \to f$ for some f. Then

$$\lim_{k\to\infty} \sin(n_k x) - \sin(n_{k+1}x) = f(x) - f(x) = 0$$

Squaring this, we get

$$\lim_{k\to\infty} \left(\sin(n_k x) - \sin(n_{k+1}x)\right)^2 = 0$$

Therefore

$$\begin{aligned}
&\lim_{k\to\infty} \int_0^{2\pi} \left(\sin(n_k x) - \sin(n_{k+1}x)\right)^2 dx \\
=&\int_0^{2\pi} \underbrace{\lim_{k\to\infty} \left(\sin(n_k x) - \sin(n_{k+1}x)\right)^2}_{0} dx \\
=&0
\end{aligned}$$

The passage of the limit inside the integral is justified by the "Bounded Convergence Theorem," since the integrand is bounded (see Chapter 11)

However, if you actually calculate the integral using double angle formulas, you get for all k

$$\int_0^{2\pi} \left(\sin(n_k x) - \sin(n_{k+1}x)\right)^2 dx = 2\pi \nrightarrow 0$$

Which is a contradiction

The result is still false even if f_n converges pointwise, see Example 7.21 in Rudin.

Note: This is sometimes stated as "The unit ball in $C[a, b]$ is not compact." That is although (f_n) is bounded, it might not have a convergent subsequence.

That said, the result **is** true if we have an extra assumption on the sequence (f_n), to make sure the sequence is well-behaved. That assumption is called:

3. Equicontinuity

Recall: f is **uniformly continuous** if for all $\epsilon > 0$ there is δ such that for all x, y, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

The point is that the δ does not depend on x, it's the same wherever we are.

Equicontinuity just means that δ doesn't depend on n, it's the same for all n:

Definition: A sequence (f_n) is (uniformly) **equicontinuous** if for all $\epsilon > 0$ there is $\delta > 0$ such that for all n and all x, y, if $|x - y| < \delta$, then $|f_n(x) - f_n(y)| < \epsilon$

Graphical Interpretation: An equicontinuous sequence is continuous in the same way, sort of like synchronous swimmers. See pictures in lecture.

Example: If (f_n') is uniformly bounded, then (f_n) is equicontinuous.

Why? Suppose $|f_n'(x)| \leq C$ for all n and all x. Let $\epsilon > 0$ be given and let $\delta = \frac{\epsilon}{C}$, then if $|x-y| < \delta$ then by the Mean-Value Theorem, we have

$$|f_n(x) - f_n(y)| = |f_n'(c)(x-y)| = |f_n'(c)|\,|x-y| \leq C\,|x-y| < C\left(\frac{\epsilon}{C}\right) = \epsilon \checkmark$$

Note: The same thing happens if (f_n) is uniformly Lipschitz:

$$|f_n(x) - f_n(y)| \leq C\,|x-y|$$

where C doesn't depend on n

What does that have to do with uniform convergence? First of all:

Theorem: If (f_n) is a sequence of functions on $C[a,b]$ that converges uniformly, then (f_n) must be equicontinuous.

Proof: Let $\epsilon > 0$ be given. Since (f_n) converges uniformly, by the Cauchy criterion, there is N such that if $n > N$, then

$$\|f_n - f_N\| < \frac{\epsilon}{3}$$

Since f_N is uniformly continuous, there is $\delta > 0$ such that if $|x-y| < \delta$ then $|f_N(x) - f_N(y)| < \frac{\epsilon}{3}$.

So if $n > N$ and $|x-y| < \delta$ then

$$\begin{aligned} |f_n(x) - f_n(y)| &\leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \checkmark \end{aligned}$$

We've only done the case for $n > N$. To extend this for *all* n, you just take the smaller one of the δ of uniform continuity of $f_1, f_2, \dots, f_N$ $\square$

4. Arzelà-Ascoli Theorem

We are ready to state and prove the celebrated Arzelà-Ascoli Theorem:

Theorem: (Arzelá-Ascoli Theorem) If (f_n) is a bounded and equicontinuous sequence in $C[a, b]$, then (f_n) has a uniformly convergent subsequence.

Note: Those kind of compactness theorems are extremely important in PDE. For example, suppose you want to solve a hard PDE. Sometimes it's easier to solve it approximately. In that case you get a sequence of approximate solutions (f_n). If the compactness result holds, you find a subsequence (f_{n_k}) that converges to some unknown function f. And if you're lucky, that limit function f solves your original PDE!

Proof:[1]

STEP 1: Fix an enumeration $\{x_1, x_2, \dots\}$ of all the rational numbers in $[a, b]$

Consider $f_n(x_1)$. This is a bounded sequence of real numbers since (f_n) is bounded, so by B-W, there is a convergent subsequence $f_{n_k}(x_1)$

Notation:

[1]The proof is taken from this Wikipedia article, as well as from Theorem 14 in Chapter 4 of Pugh's book

$$\begin{aligned} f_{0,k} =& f_k \text{ original sequence} \\ f_{1,k} =& f_{n_k} \text{ subsequence} \\ f_{2,k} =& \text{ sub-subsequence (see below)} \\ f_{m,k} =& \text{ sub-sub... sequence} \end{aligned}$$

m is the "depth" of the sequence, and k is the term of the sequence

Since $f_{1,k}(x_2)$ is bounded, there is a sub-subsequence $f_{2,k}$ such that $f_{2,k}(x_2)$ converges. Notice $f_{2,k}$ converges as x_1 as well. So $f_{2,k}$ converges at x_1 and x_2

That way we obtain a tower of subsequences

$$f_n \supseteq f_{1,k} \supseteq f_{2,k} \supseteq \cdots$$

Such that $f_{m,k}$ converges at $x_1, x_2, \ldots, x_m$ (see the picture in lecture)

STEP 2: Consider the diagonal subsequence $g_m =: f_{m,m}$, which is the m-th term of the m-th subsequence.

By construction, g_m converges at every rational point.

Claim: (g_m) converges uniformly.

Then we would be done because then (g_m) is a subsequence of (f_n) that converges uniformly.

STEP 3: **Proof of Claim:** We will show that (g_m) is Cauchy.

Here is where equicontinuity kicks in:

Let $\epsilon > 0$ be given.

By equicontinuity there is $\delta > 0$ such that for all x, y and all m:

$$|x - y| < \delta \Rightarrow |g_m(x) - g_m(y)| < \frac{\epsilon}{3}$$

Intuitively: Rational points are good (because g_m converges on them) and δ is good (because of continuity), it makes sense to cover $[a, b]$ with balls centered at rational points and radius δ:

Consider the balls (intervals) $B(x_1, \delta), B(x_2, \delta), \dots$ They cover $[a, b]$ so by compactness there is a finite sub-cover, which we'll relabel as $B(x_1, \delta), B(x_2, \delta), \dots, B(x_I, \delta)$.

Since $g_m(x_i)$ converges for each x_i as above, it is Cauchy, so there is N such that for all $m, n > N$ and all $i = 1, 2, \dots, I$

$$|g_m(x_i) - g_n(x_i)| < \frac{\epsilon}{3}$$

Now we're ready to conclude!

With the same N, if $m, n > N$ and $x \in [a, b]$, choose x_i as above such that $|x_i - x| < \delta$ (can do that by def of a cover) then

$$\begin{aligned} |g_m(x) - g_n(x)| &\leq |g_m(x) - g_m(x_i)| + |g_m(x_i) - g_n(x_i)| + |g_n(x_i) - g_n(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \checkmark \end{aligned}$$

(By equicontinuity, Cauchiness, and equicontinuity)

This is all I have to say about uniform convergence! The next 3 mini-topics have more to do with functions in general.