LECTURE 30: THE JACOBIAN (II)

1. More Practice

Sometimes the change of variables is in the region D instead of the function:

Example 1:

$$\int\int_D y\,dxdy$$

D is the region between $xy = 1, xy = 2, xy^2 = 3, xy^2 = 5$

STEP 1:

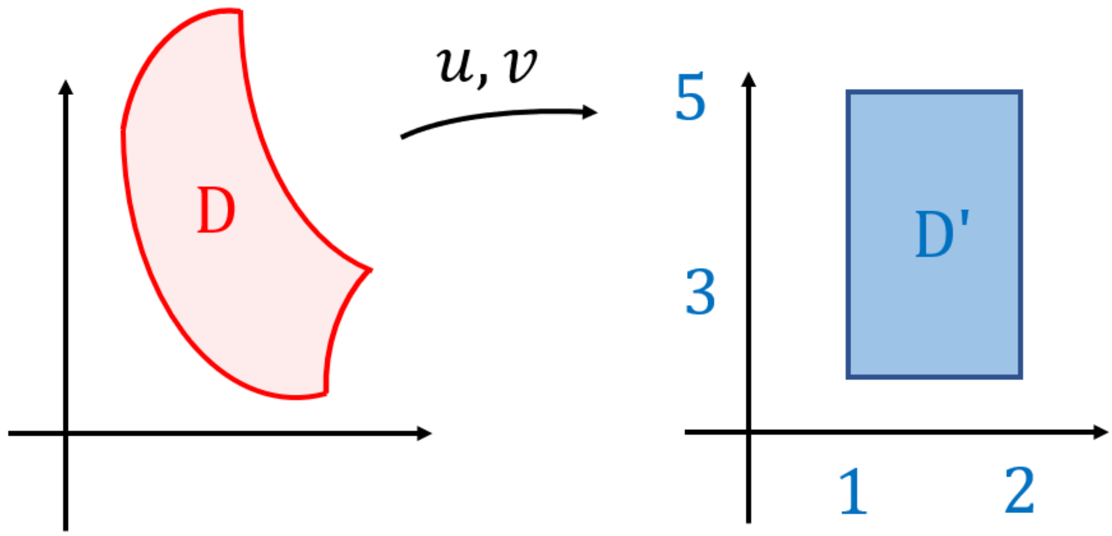
$$\begin{cases} u = xy \\ v = xy^2 \end{cases}$$

STEP 2: **Endpoints**

Note: You don't know what D looks like, but it doesn't really matter!

$$\begin{cases} 1 \leq xy \leq 2 \\ 3 \leq xy^2 \leq 5 \end{cases} \Rightarrow \begin{cases} 1 \leq u \leq 2 \\ 3 \leq v \leq 5 \end{cases}$$

Date: Friday, November 5, 2021.



STEP 3: Jacobian

$$dudv = \left|\frac{dudv}{dxdy}\right| dxdy \qquad u = xy, v = xy^2$$

$$\frac{dudv}{dxdy} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ y^2 & 2xy \end{vmatrix} = y(2xy) - xy^2 = 2xy^2 - xy^2 = xy^2$$

$$dudv = \left|xy^2\right| dxdy = xy^2\, dxdy = v\, dxdy \Rightarrow dxdy = \frac{1}{v} dudv$$

STEP 4: Integrate:

Here $f(x,y) = y$, but notice that:

$$v = xy^2 = (xy)\, y = uy \Rightarrow y = \frac{v}{u}$$

$$\begin{aligned}
&\int\int_D y dx dy \\
=&\int\int_{D'}\left(\frac{v}{u}\right)\left(\frac{1}{v} du dv\right) \\
=&\int_3^5\int_1^2 \frac{1}{u} du dv \\
=&(5-3)\left[\ln|u|\right]_1^2 \\
=&2\left(\ln(2)-\ln(1)\right) \\
=&2\ln(2)
\end{aligned}$$

2. The Reverse Way

Sometimes, you have to do $u-$sub in reverse

Example 2:

$$\int\int_D x - 2y \, dx dy$$

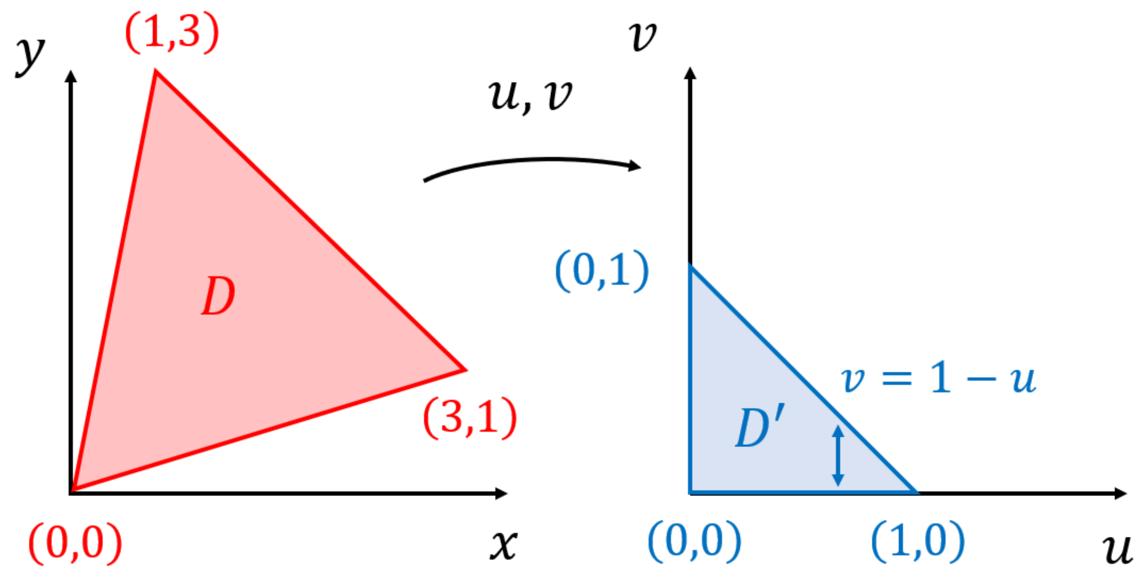
Where D is the triangle with vertices $(0,0),(1,3),(3,1)$

STEP 1: Suppose someone tells you (this **WILL** be given)

$$\begin{cases} x = 3u + v \\ y = u + 3v \end{cases}$$

STEP 2: Endpoints

Again, look at the vertices:



$$(0,0) \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \Rightarrow \begin{cases} 3u + v = 0 \\ u + 3v = 0 \end{cases} \Rightarrow \begin{cases} u = 0 \\ v = 0 \end{cases} \Rightarrow (0,0)$$

$$(1,3) \Rightarrow \begin{cases} x = 1 \\ y = 3 \end{cases} \Rightarrow \begin{cases} 3u + v = 1 \\ u + 3v = 3 \end{cases} \Rightarrow \begin{cases} v = 1 - 3u \\ u + 3(1 - 3u) = 3 \end{cases} \Rightarrow \begin{cases} v = 1 \\ u = 0 \end{cases}$$

So $(1,3)$ becomes $(0,1)$. And similarly, $(3,1)$ becomes $(1,0)$

So D' is an (easier) triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$

STEP 3: Jacobian:

$$dxdy = \left| \frac{dxdy}{dudv} \right| dudv = |8|\, dudv = 8dudv$$

$$\frac{dxdy}{dudv} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 9 - 1 = 8$$

(Here we used $x = 3u + v, y = u + 3v$)

STEP 4: Integrate:

$$f(x, y) = x - 2y = (3u + v) - 2(u + 3v) = u - 5v$$

$$\begin{aligned}
&\int\int_D x - 2y \, dxdy \\
=&\int\int_{D'} (u - 5v) \, 8 \, dudv \\
=&8 \int_0^1 \int_0^{1-u} u - 5v \, dvdu \\
=&8 \int_0^1 \left[uv - \frac{5}{2} v^2 \right]_{v=0}^{v=1-u} du \\
=&8 \int_0^1 u(1-u) - \frac{5}{2}(1-u)^2 \, du \\
=&8 \int_0^1 u - u^2 - \frac{5}{2}(u-1)^2 \, du \\
=&8 \left[\frac{u^2}{2} - \frac{u^3}{3} - \frac{5}{6}(u-1)^3 \right]_0^1 \\
=&8 \left(\frac{1}{2} - \frac{1}{3} - \frac{5}{6}(1-1)^3 + \frac{5}{6}(0-1)^3 \right) \\
=&8 \left(\frac{1}{6} - \frac{5}{6} \right) \\
=&8 \left(-\frac{2}{3} \right) \\
=&-\frac{16}{3}
\end{aligned}$$

Note: For an extra practice problem, check out this video:

Video: Change of Variables

3. Polar Coordinates

Video: The Jacobian 2

As an application, let's show why, in polar coordinates, we have r in $rdrd\theta$:

Example 3:

$$\int\int_D \tan^{-1}\left(\frac{y}{x}\right)\, dxdy$$

D = Disk of Radius 1.

STEP 1:

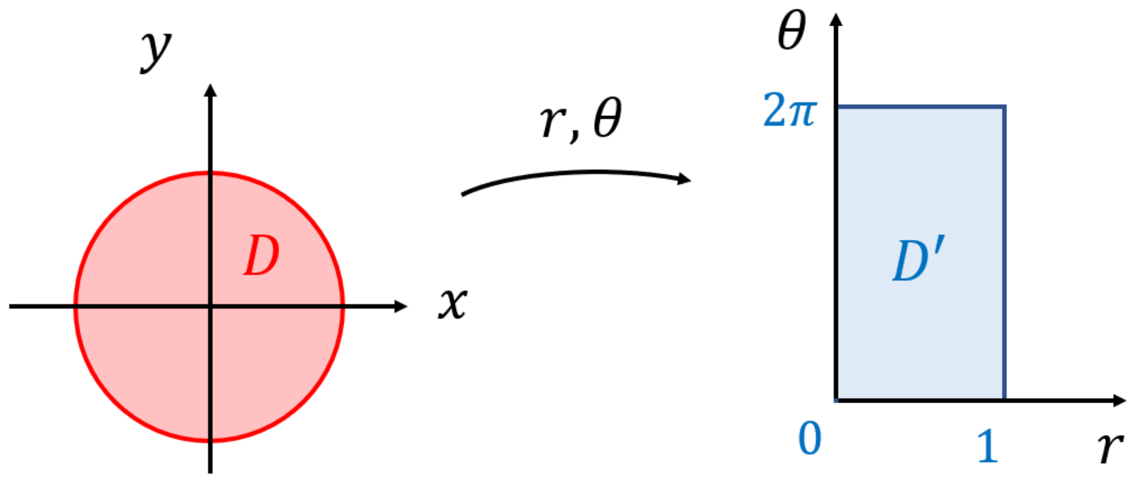
$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

Here r and θ play the role of u and v.

STEP 2: Endpoints:

$$\begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

So r and θ effectively turn the disk D into a rectangle D'



STEP 3: **Jacobian**

$$dxdy = \left|\frac{dxdy}{drd\theta}\right| drd\theta = |r|\, drd\theta = r drd\theta$$

$$\frac{dxdy}{drd\theta} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r$$

STEP 4: **Integrate**

$$
\begin{aligned}
&\int\int_{D} \tan^{-1}\left(\frac{y}{x}\right) dxdy \\
=&\int\int_{D'} \theta r \, dr d\theta \\
=&\int_0^{2\pi}\int_0^1 \theta r dr d\theta \\
=&\left(\int_0^1 r dr\right)\left(\int_0^{2\pi} \theta d\theta\right) \\
=&\left[\frac{r^2}{2}\right]_0^1 \left[\frac{\theta^2}{2}\right]_0^{2\pi} \\
=&\left(\frac{1}{2}\right)\left(\frac{4\pi^2}{2}\right) \\
=&\pi^2
\end{aligned}
$$

Note: Similarly, for cylindrical coordinates, we get $dxdydz = rdrd\theta dz$, and in spherical coordinates we get $d\rho d\theta d\phi = \rho^2 \sin(\phi)\, d\rho d\theta d\phi$ (see appendix below)

4. As easy as $\frac{4}{3}\pi abc$

In this final application, we'll find a super easy way of calculating the volume of an ellipsoid

Example 4:

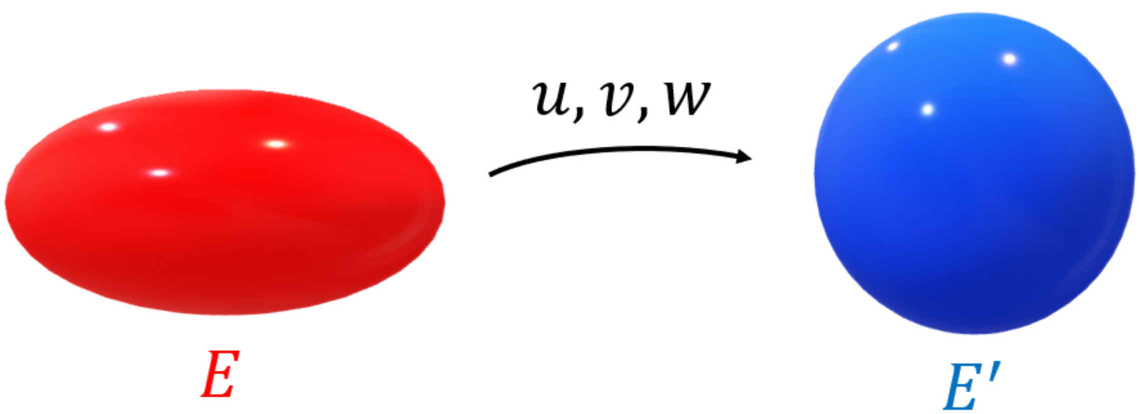
Find the volume of the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1$$

STEP 1: $u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}$

STEP 2: Endpoints

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1 \Rightarrow u^2 + v^2 + w^2 \leq 1 \text{ Ball of radius 1}$$



STEP 3:

$$dudvdw = \left|\frac{dudvdw}{dxdydz}\right| dxdydz \qquad u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}$$

$$\frac{dudvdw}{dxdydz} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{vmatrix} = \frac{1}{abc}$$

$$dudvdw = \left|\frac{1}{abc}\right| dxdydz = \frac{1}{abc} dxdydz \Rightarrow dxdydz = abc\, dudvdw$$

STEP 4:

$$
\begin{aligned}
\text{Vol}(E) &= \int\int\int_E 1 \, dxdydz \\
&= \int\int\int_{E'} 1abc \, dudvdw \\
&= abc \int\int\int_{E'} dudvdw \\
&= abc \ \text{Vol}(E') \\
&= abc \, \frac{4}{3}\pi \left(1\right)^3 \quad (E' \text{ is a ball of radius } 1) \\
&= \frac{4}{3}\pi abc
\end{aligned}
$$

Finally, for an unbelievably cool application of Jacobians, check out the following optional video:

Optional Video: The Jacobian 3

5. Appendix: Spherical Coordinates

Similarly to what we did with polar coordinates, we can derive the $\rho^2 \sin(\phi)$ term for spherical coordinates

Example 5:

$$\int\int\int_E \sqrt{x^2 + y^2 + z^2} dxdydz$$

E : Ball of radius 1

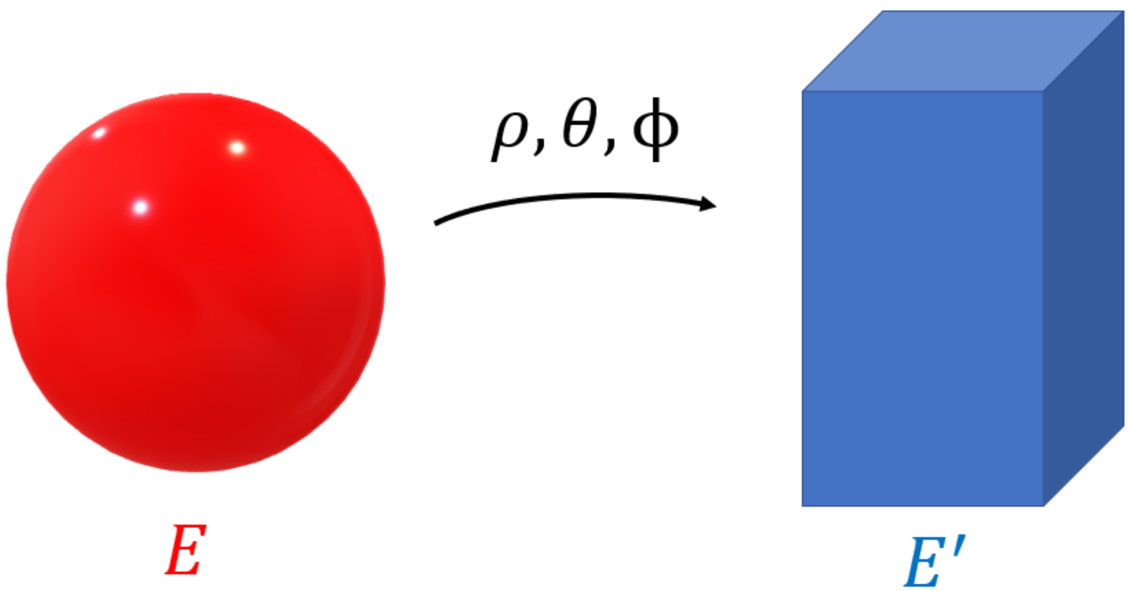
STEP 1:

$$\begin{cases} x = & \rho \sin(\phi) \cos(\theta) \\ y = & \rho \sin(\phi) \sin(\theta) \\ z = & \rho \cos(\phi) \end{cases}$$

STEP 2: Endpoints

$$0 \leq \rho \leq 1$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \phi \leq \pi$$

So here E' is a box



STEP 3: Jacobian

$$dxdydz = \left| \frac{dxdydz}{d\rho d\theta d\phi} \right| d\rho d\theta d\phi$$

Here $x = \rho \sin(\phi) \cos(\theta), y = \rho \sin(\phi) \sin(\theta), z = \rho \cos(\phi)$, so

$$
\begin{aligned}
\frac{dxdydz}{d\rho d\theta d\phi} &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} \\
&= \begin{vmatrix} \sin(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) & \rho\cos(\phi)\sin(\theta) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{vmatrix} \\
&= \cos(\phi) \begin{vmatrix} -\rho\sin(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \rho\sin(\phi)\cos(\theta) & \rho\cos(\phi)\sin(\theta) \end{vmatrix} \\
&\quad - 0 \text{ (Blah)} \\
&\quad - \rho\sin(\phi) \begin{vmatrix} \sin(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) \end{vmatrix} \\
&= \cos(\phi)\left(-\rho^2\cos(\phi)\sin(\phi)\sin^2(\theta) - \rho^2\sin(\phi)\cos(\phi)\cos^2(\theta)\right) \\
&\quad - \rho\sin(\phi)\left(\rho\sin^2(\phi)\cos^2(\theta) + \rho\sin^2(\phi)\sin^2(\theta)\right) \\
&= \cos(\phi)\left(-\rho^2\cos(\phi)\sin(\phi)\right) - \rho\sin(\phi)\left(\rho\sin^2(\phi)\right) \\
&= -\rho^2\sin(\phi)\cos^2(\phi) - \rho^2\sin(\phi)\sin^2(\phi) \\
&= -\rho^2\sin(\phi)
\end{aligned}
$$

$$
dxdydz = \left|-\rho^2\sin(\phi)\right| d\rho d\theta d\phi = \rho^2\sin(\phi) d\rho d\theta d\phi
$$

STEP 4: Integrate

$$
\begin{aligned}
&\int\int\int_E \sqrt{x^2+y^2+z^2}\, dxdydz \\
&= \int\int\int_{E'} \rho\left(\rho^2\sin(\phi)\right) d\rho d\theta d\phi \\
&= \int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^3\sin(\phi) d\rho d\theta d\phi
\end{aligned}
$$

$$= \left(\int_0^1 \rho^3 d\rho \right) (2\pi) \left(\int_0^\pi \sin(\phi) d\phi \right)$$

$$= \left(\frac{1}{4} \right) (2\pi) (2)$$

$$= \pi$$