LECTURE 31: VECTOR FIELDS

Welcome to the final chapter of your final calculus course! Like all boss battles, this chapter is unbelievably hard, but also unbelievably exciting!

Today: A very gentle introduction to vector fields, just to show you how awesome they are!

1. Vector Fields

Definition:

A **vector field F** is a function that associates to each point (x, y) a vector $\mathbf{F}(x, y)$

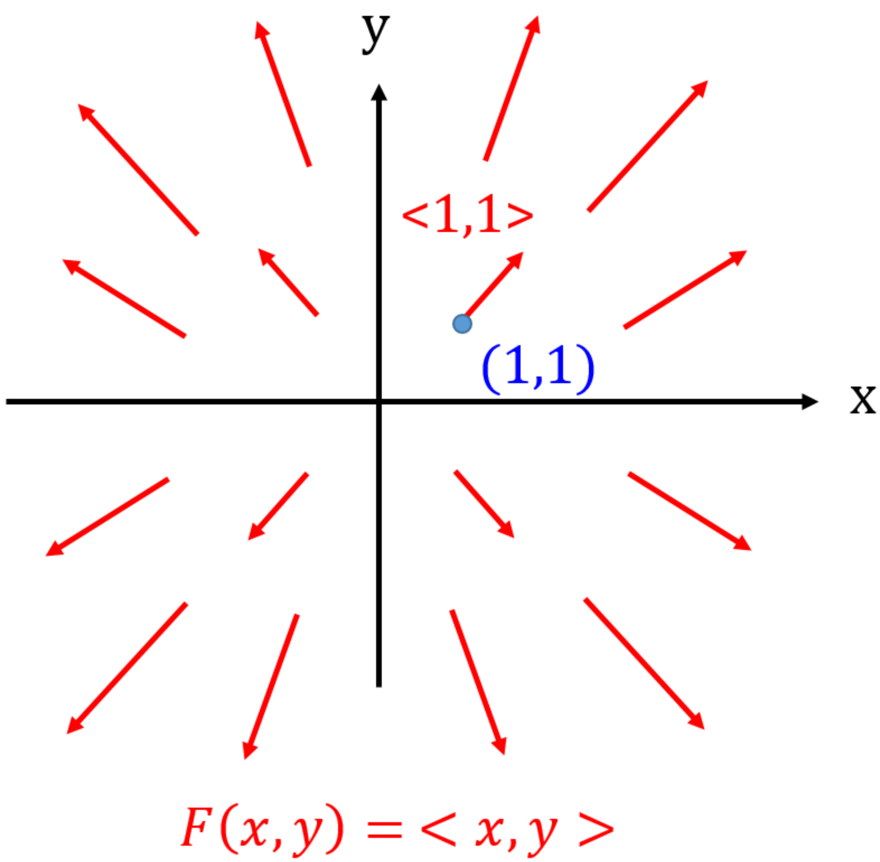
Really abstract definition, but a picture says 1000 words:

Example 1:

Sketch $\mathbf{F}(x, y) = \langle x, y \rangle$

$$\mathbf{F}(1,1) = \langle 1, 1 \rangle$$
$$\mathbf{F}(3,1) = \langle 3, 1 \rangle$$
$$\mathbf{F}(2,2) = \langle 2, 2 \rangle$$
$$\mathbf{F}(-3,1) = \langle -3, 1 \rangle$$

Date: Monday, November 8, 2021.



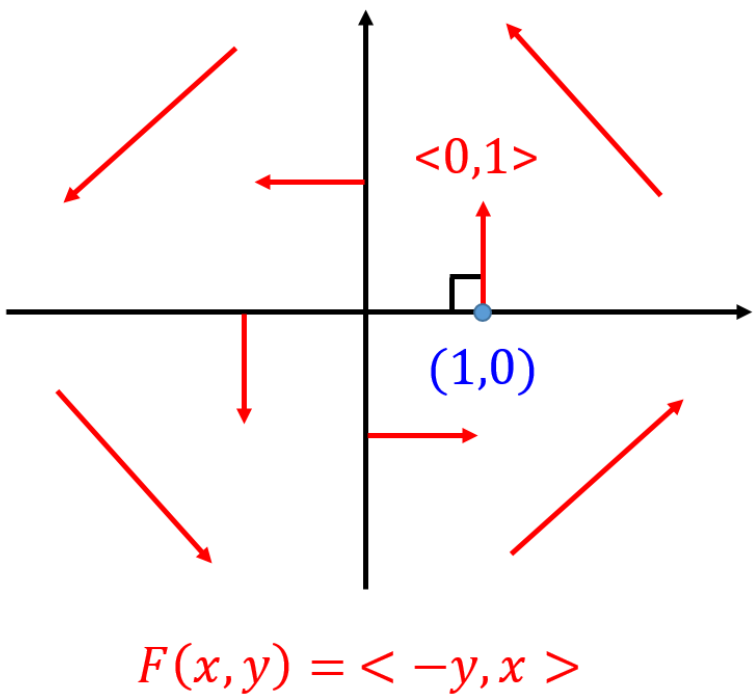
So a vector field is just a bunch of vectors.

Many applications: Force Field, Velocity Field, Gravitational Field, Electrostatic Field, Emotional Attraction...

Example 2:

Sketch $\mathbf{F}(x, y) = \langle -y, x\rangle$

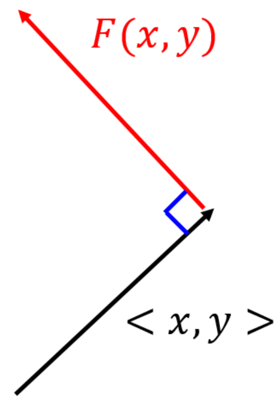
$$\begin{aligned}\mathbf{F}(1,0) &= \langle -0, 1\rangle = \langle 0, 1\rangle \\ \mathbf{F}(0,1) &= \langle -1, 0\rangle \\ \mathbf{F}(-1,0) &= \langle 0, -1\rangle \\ \mathbf{F}(0,-1) &= \langle 1, 0\rangle\end{aligned}$$



Note: Indeed, we have

$$\mathbf{F}(x,y)\cdot\langle x,y\rangle = \langle -y,x\rangle\cdot\langle x,y\rangle = -yx+xy = 0$$

So $\mathbf{F}(x,y)$ is perpendicular to $\langle x,y\rangle$



Alternate notation:

$$\mathbf{F}(x, y) = \langle -y, x \rangle = -y\mathbf{i} + x\mathbf{j}$$

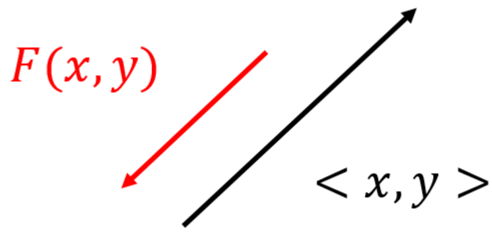
Where $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$

Example 3:

Sketch the following vector field:

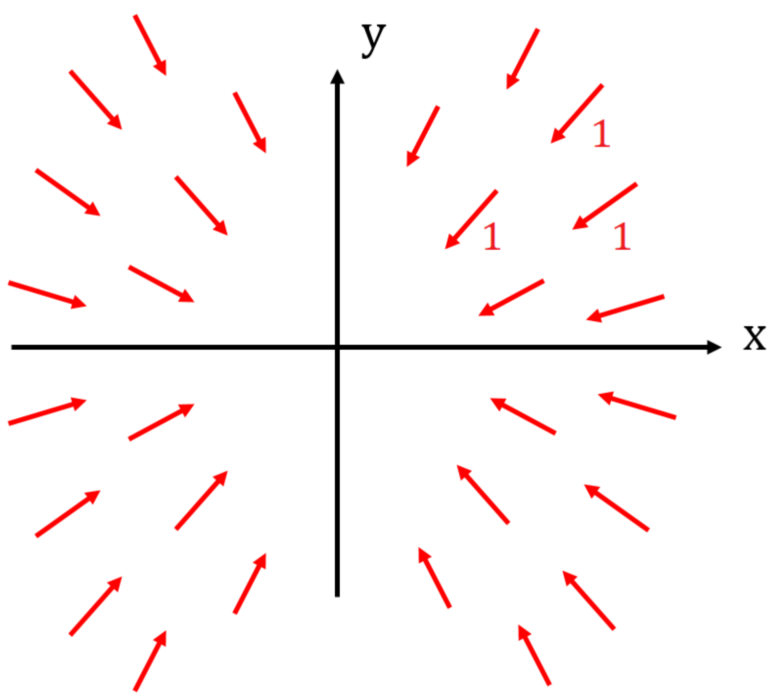
$$\mathbf{F}(x, y) = \left(-\frac{x}{\sqrt{x^2 + y^2}}\right)\mathbf{i} + \left(-\frac{y}{\sqrt{x^2 + y^2}}\right)\mathbf{j}$$
$$= \left\langle -\frac{x}{\sqrt{x^2 + y^2}}, -\frac{y}{\sqrt{x^2 + y^2}} \right\rangle$$

$\mathbf{F}(x, y) = \left(-\frac{1}{\sqrt{x^2+y^2}}\right)\langle x, y \rangle$, hence it points the opposite direction from $\langle x, y \rangle$



$$\|\mathbf{F}(x,y)\| = \sqrt{\left(-\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(-\frac{y}{\sqrt{x^2+y^2}}\right)^2} = \sqrt{\frac{x^2+y^2}{x^2+y^2}} = \sqrt{1} = 1$$

Hence $\mathbf{F}(x,y)$ has length 1



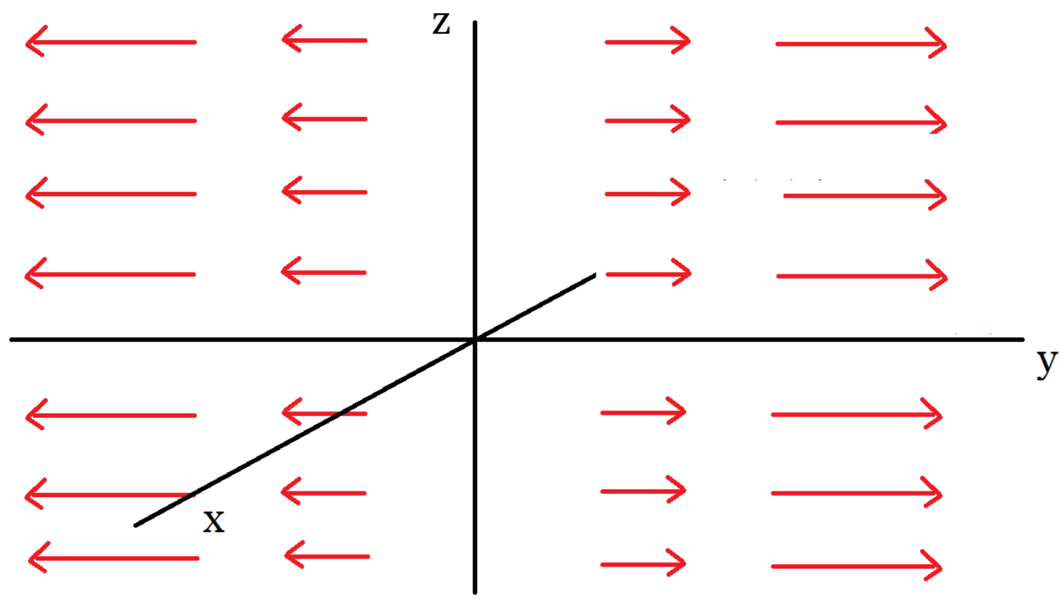
Kind of looks like gravity/black hole!

Of course we can do all this in 3 dimensions as well:

Example 4:

Sketch $\mathbf{F}(x, y, z) = \langle 0, y, 0 \rangle$

Here $\mathbf{F}$ independent of x and z, and proportional to $\langle 0, 1, 0 \rangle$



Point: Everything you learn in 2 dimensions can be generalized to 3 dimensions and beyond!

2. Gradient Fields

Given a function $f(x, y)$, we can create a nice vector field associated to f called the **gradient field**:

Definition:

If $f(x, y)$ is a function, then

$$\mathbf{F} = \nabla f = \langle f_x, f_y \rangle$$

is called the **gradient field** of f.

Example 5:

The gradient field of $f(x, y) = x^2 y - y^3$ is:

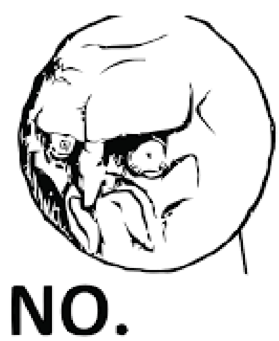
$$\mathbf{F} = \nabla f = \left\langle (x^2 y - y^3)_x, (x^2 y - y^3)_y \right\rangle = \left\langle 2xy, x^2 - 3y^2 \right\rangle$$

Notice that $\mathbf{F}$ is indeed a vector field! It's a *nice* vector field that you associate to a function f

Big Question:

Are all vector fields $\mathbf{F}$ gradient fields? In other words, can all vector fields $\mathbf{F}$ be written in the form $F = \nabla f$ for some f?

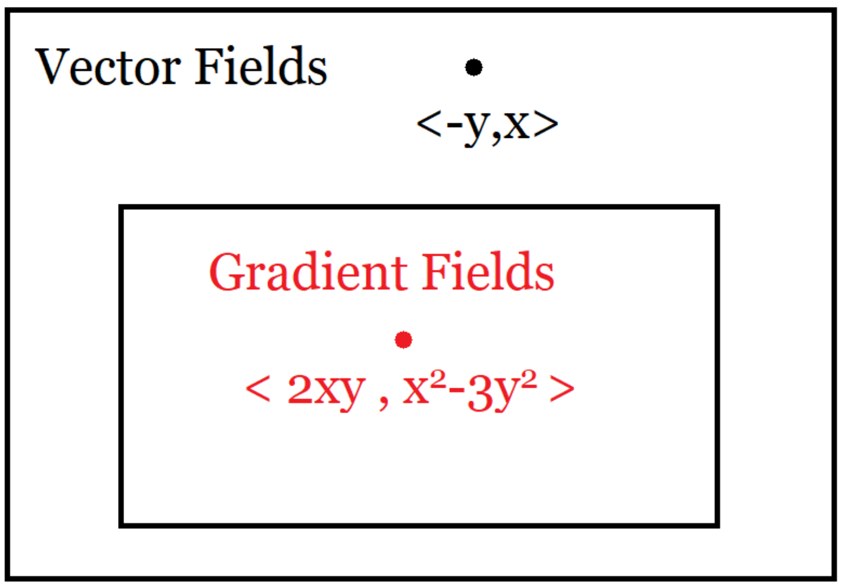
Answer:



Non-Example 6:

$\mathbf{F}(x, y) = \langle -y, x \rangle$ (rotation field), then F cannot be written as ∇f for any f (see later why)

This is quite surprising, because even though every function has an antiderivative, not every vector field $\mathbf{F}$ has an antiderivative!



But **if** $\mathbf{F}$ is a gradient field, we call this ~~nice~~ conservative[1]

Definition:

$\mathbf{F}$ is **conservative** if $\mathbf{F} = \nabla f$ for some f

Note: f is called a **potential function**, useful in physics.

[1] Conservative because of conservation of energy, not because of political preferences ☺

Example 7:

$\mathbf{F}(x,y) = \left\langle 2xy, x^2 - 3y^2 \right\rangle$ is conservative because $\mathbf{F} = \nabla f$ for $f(x,y) = x^2y - y^3$

Example 8:

$$\mathbf{F}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

Is conservative, because if $f(x,y) = \sqrt{x^2+y^2}$, then:

$$\begin{aligned}
\nabla f &= \langle f_x, f_y \rangle \\
&= \left\langle \left(\sqrt{x^2+y^2}\right)_x, \left(\sqrt{x^2+y^2}\right)_y \right\rangle \\
&= \left\langle \frac{2x}{2\sqrt{x^2+y^2}}, \frac{2y}{2\sqrt{x^2+y^2}} \right\rangle \\
&= \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle \\
&= \mathbf{F} \qquad \checkmark
\end{aligned}$$

Goals of this chapter:

(1) What makes conservative vector fields so nice?

(2) How to determine **F** is conservative or not

Note: Check out the following video for a pretty good overview of the topics in this chapter:

Video: Vector Calculus Overview

3. Pretty Pictures

This chapter is incomplete without pretty pictures! Check out this Reddit post for pretty examples of vector fields:

Reddit: Pretty Pictures (start with "A Neat One")