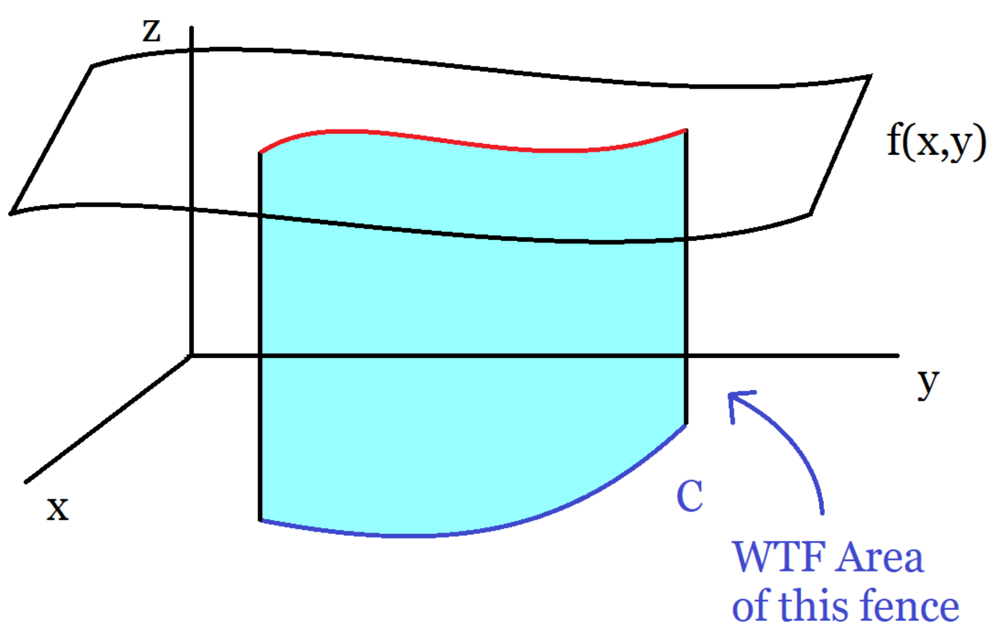
LECTURE 32: LINE INTEGRALS (I)

Today: We'll do something really cool: we'll integrate a function over a curve!

1. Line Integrals

Video: Line Integral

Goal: Given a curve C and a function $f(x, y)$, find the area of the fence under f and over C



Date: Wednesday, November 10, 2021.

Notation:

$$\int_C f(x,y)ds = \textit{Line} \text{ Integral of } f \text{ over } C$$

Example 1:

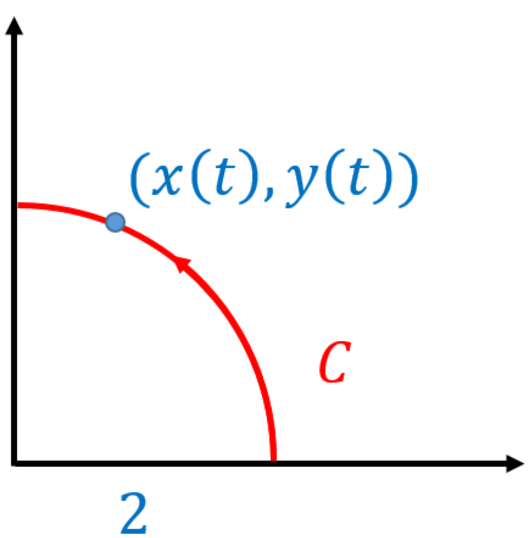
$$\int_C x^2 y \, ds$$

C : Quarter circle $x^2+y^2=4$ in first quadrant (counterclockwise)

Note: The key for today's problems is parametrization; for a review of parametrizations, check out the following video:

Video: Parametric Equations

STEP 1: Picture:



STEP 2: Parametrize C:

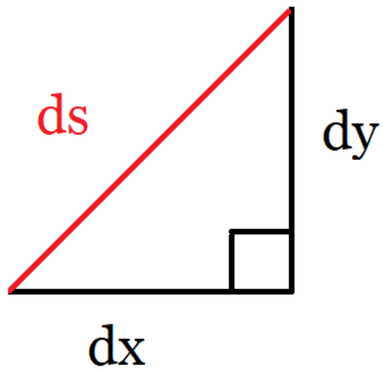
$$\begin{cases} x(t) = 2\cos(t) \\ y(t) = 2\sin(t) \quad \left(0 \leq t \leq \frac{\pi}{2}\right) \end{cases}$$

STEP 3: Integrate: Use following fact:

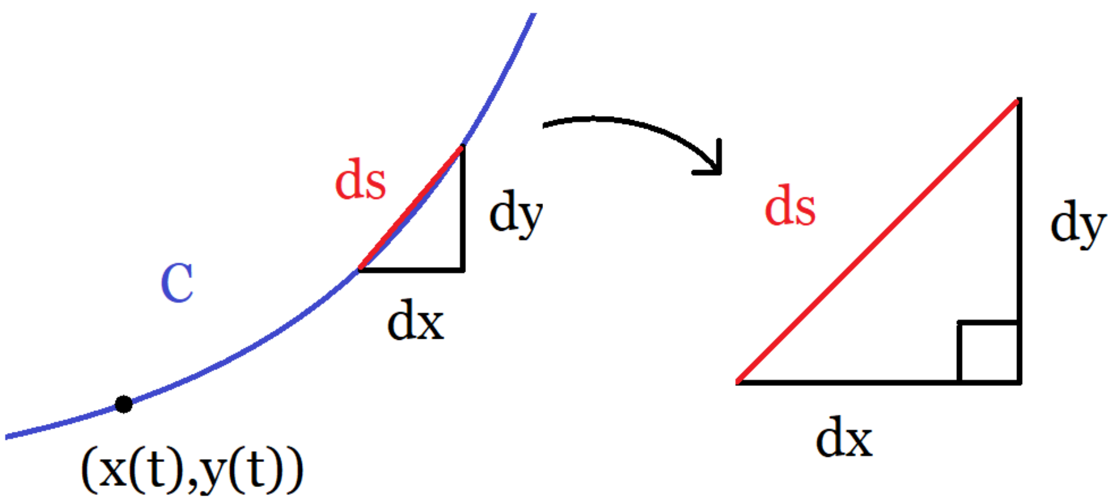
$$\boxed{ds = \sqrt{(x'(t))^2 + (y'(t))^2} dt}$$

$$\begin{aligned}
&\int_C x^2 y \, ds \\
=&\int_0^{\frac{\pi}{2}} (x(t))^2 (y(t)) \sqrt{(x'(t))^2 + (y'(t))^2} \, dt \\
=&\int_0^{\frac{\pi}{2}} (2\cos(t))^2 \, 2\sin(t) \sqrt{(-2\sin(t))^2 + (2\cos(t))^2} dt \\
=&\int_0^{\frac{\pi}{2}} 8\cos^2(t)\sin(t) \underbrace{\sqrt{4\sin^2(t) + 4\cos^2(t)}}_{2} \, dt \\
=&\int_0^{\frac{\pi}{2}} 8\cos^2(t)\sin(t) \, 2dt \qquad (u = \cos(t)) \\
=&16\left[-\frac{1}{3}\cos^3(t)\right]_0^{\frac{\pi}{2}} \\
=&\frac{16}{3}
\end{aligned}$$

Mnemonic: $ds = \sqrt{(dx)^2 + (dy)^2}$ (kind of like a diagonal length)



In fact, this is how ds is derived: you approximate C with diagonal segments, each having length ds, you multiply by f and then you integrate.



The details are in the appendix, as well as the video below. Note that have actually already derived this result back in the section on arclength.

Video: Line Integral Derivation

Applications:

- $\int_C f$ = Area of fence under f over C
- If f = Density, then $\int_C f ds$ = Mass of (wire) C
- $\int_C 1\, ds$ = Length of C (compare to arclength formula)

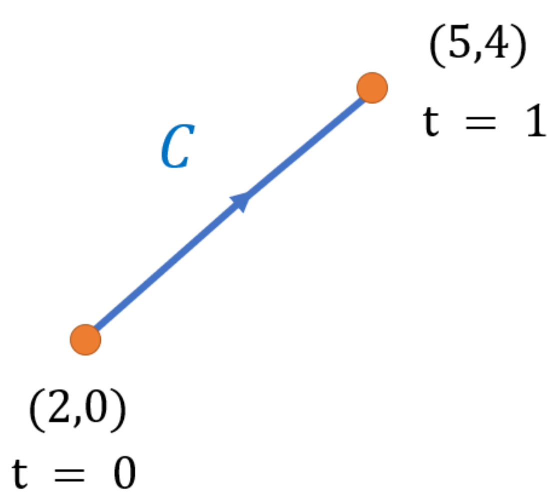
Video: Line Integral Example

Example 2:

Find the mass of the wire C with density $f(x, y) = y$, where C is the line from $(2, 0)$ to $(5, 4)$

(The higher up we go, the heavier the wire is)

STEP 1: Picture:



STEP 2: Parametrize:

Here we have the line connecting $(2,0)$ and $(5,4)$, so

$$\begin{cases} x(t) = (1-t)\,2 + t\,(5) = 3t+2 \\ y(t) = (1-t)\,0 + t\,(4) = 4t \\ \qquad (0 \leq t \leq 1) \end{cases}$$

Note: In case you're having doubts about whether to put t or $1-t$, simply check that at $t=0$ you get $(2,0)$ and at $t=1$ you get $(5,4)$

STEP 3: Integrate: Here $f(x,y) = y$

$$\begin{aligned} &\int_C y\, ds \\ =&\int_0^1 y(t)\sqrt{(x'(t))^2 + (y'(t))^2}dt \\ =&\int_0^1 4t\sqrt{3^2+4^2}dt \\ =&\int_0^1 4t(5) \\ =&\int_0^1 20t \\ =&10 \end{aligned}$$

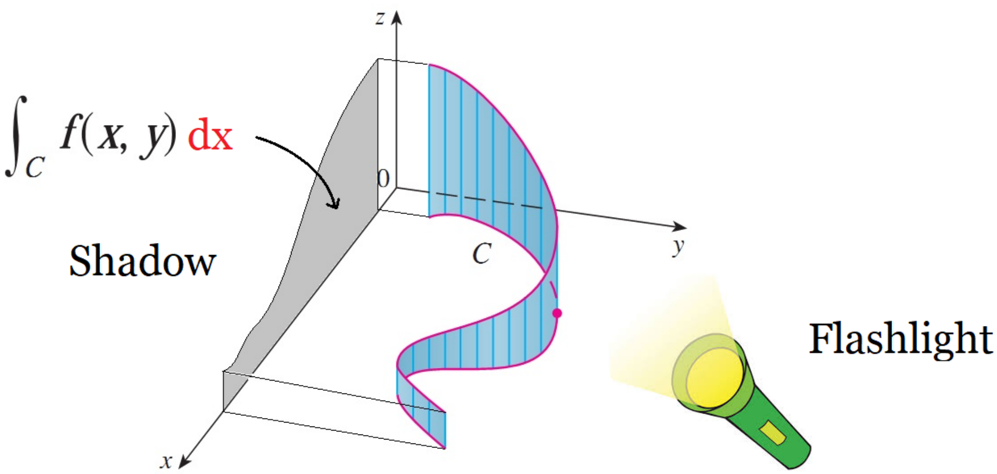
Note: For even more practice, you can check out the video below:

Video: Line Integral of a Function

2. Two Variations

Video: Line integral with respect to x

Two variations on this idea: Suppose you have a flashlight in the y-direction, and you cast a light on the fence, then you get a shadow on the left [1]



$$\text{Area of Left Shadow } = \int_C f(x,y)dx = \int_a^b f(x,y)\frac{dx}{dt}dt = \int_a^b f(x(t),y(t))x'(t)dt$$

Similarly, if you have a flashlight in the $x-$direction:

$$\text{Area of Back Shadow } = \int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt$$

Can even combine the two: (For now think in terms of shadows, but we'll find a better interpretation next time)

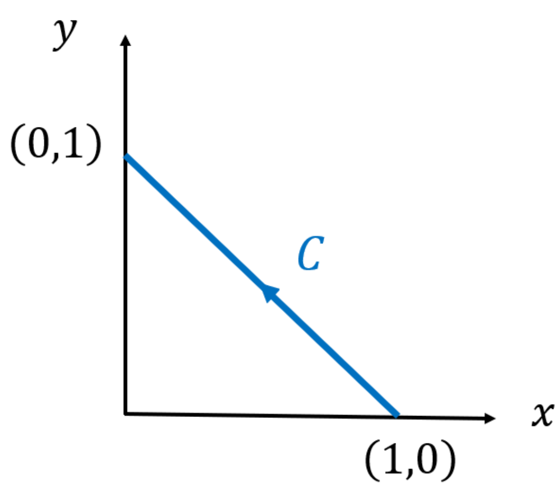
[1] Source: Math Stackexchange

Example 3:

Calculate $\int_C -ydx + xdy$, where

(a) C is the line segment from $(1,0)$ to $(0,1)$

STEP 1: Picture



STEP 2: Parametrize

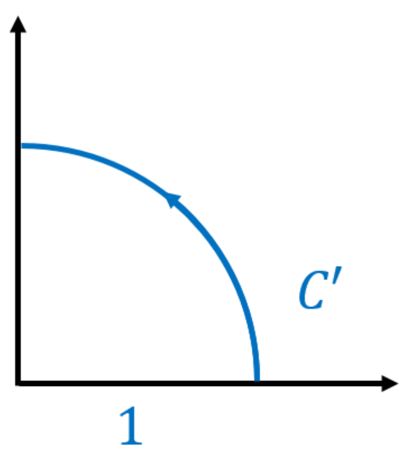
$$\begin{cases} x(t) = (1-t)\,1 + t\,(0) = 1-t \\ y(t) = (1-t)\,0 + t\,(1) = t \\ \quad (0 \leq t \leq 1) \end{cases}$$

STEP 3: Integrate

$$
\begin{aligned}
&\int_C -y\,dx + x\,dy \\
=&\int_0^1 -y(t)x'(t) + x(t)y'(t)\,dt \quad \text{(By definition)} \\
=&\int_0^1 -t(-1) + (1-t)(1)dt \\
=&\int_0^1 1dt \\
=&1
\end{aligned}
$$

(b) C' is the quarter circle from $(1,0)$ to $(0,1)$

STEP 1: Picture



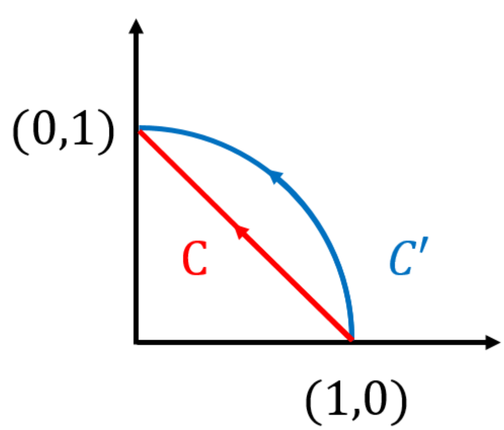
STEP 2: Parametrize

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \\ \left(0 \leq t \leq \frac{\pi}{2}\right) \end{cases}$$

STEP 3: Integrate

$$\begin{aligned} &\int_{C'} -y dx + x dy \\ =&\int_0^{\frac{\pi}{2}} -y(t)x'(t) + x(t)y'(t)\, dt \\ =&\int_0^{\frac{\pi}{2}} -\sin(t)\left(-\sin(t)\right) + \cos(t)\cos(t) dt \\ =&\int_0^{\frac{\pi}{2}} 1 dt \\ =&\frac{\pi}{2} \end{aligned}$$

Remark: Even though C and C' have the same endpoints, we get two different answers.



In this case, we say that the line integral **depends** on the path.

Big Question:

When is the line integral *independent* of the path?

Surprisingly, this has to do with…conservative vector fields!!

3. More practice: 3 dimensions

Video: Integral over a helix

Of course, everything we learn today can be generalized to 3D:

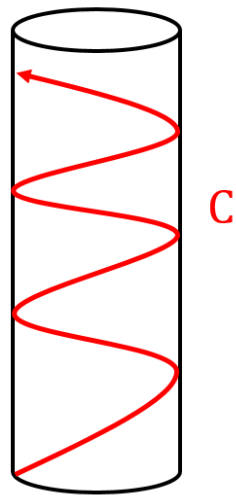
Example 4: (more practice)

$$\int_C x^2 + y^2 + z^2 \, ds$$

C : Helix parametrized by

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \\ z(t) = t \\ \quad (0 \leq t \leq 6\pi) \end{cases}$$

STEP 1: Picture:



STEP 2: Parametrize: $\checkmark$

STEP 3: Integrate:

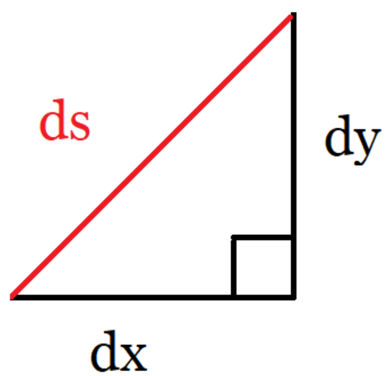
$$
\begin{aligned}
&\int_C x^2 + y^2 + z^2 \, ds \\
=&\int_0^{6\pi} \left((x(t))^2 + (y(t))^2 + (z(t))^2\right) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt \\
=&\int_0^{6\pi} \left(\cos^2(t) + \sin^2(t) + t^2\right) \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2} dt \\
=&\int_0^{6\pi} \left(1 + t^2\right) \sqrt{2} dt \\
=&\sqrt{2}\left(6\pi + \frac{(6\pi)^3}{3}\right)
\end{aligned}
$$

4. Appendix: What is ds ?

Video: Line Integral Derivation

In this appendix, we'll derive the formula for the line integral $\int_C f ds$

Idea: If C is a curve and you change x and y a little bit to get dx and dy, then ds is just the change in diagonal (which is close to C)



To find ds, just use the Pythagorean Theorem

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

Important Trick:

$$dx = \frac{dx}{dt}dt = x'(t)dt$$
$$dy = \frac{dy}{dt}dt = y'(t)dt$$

So ds is just

$$
\begin{aligned}
ds = \sqrt{(dx)^2 + (dy)^2} &= \sqrt{(x'(t)dt)^2 + (y'(t)dt)^2} \\
&= \sqrt{(x'(t))^2(dt)^2 + (y'(t))^2(dt)^2} \\
&= \sqrt{(x'(t))^2 + (y'(t))^2}\, dt
\end{aligned}
$$

Finally, to get $\int_C f$, just multiply the above by f and integrate:

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{(x'(t))^2 + (y'(t))^2}dt$$