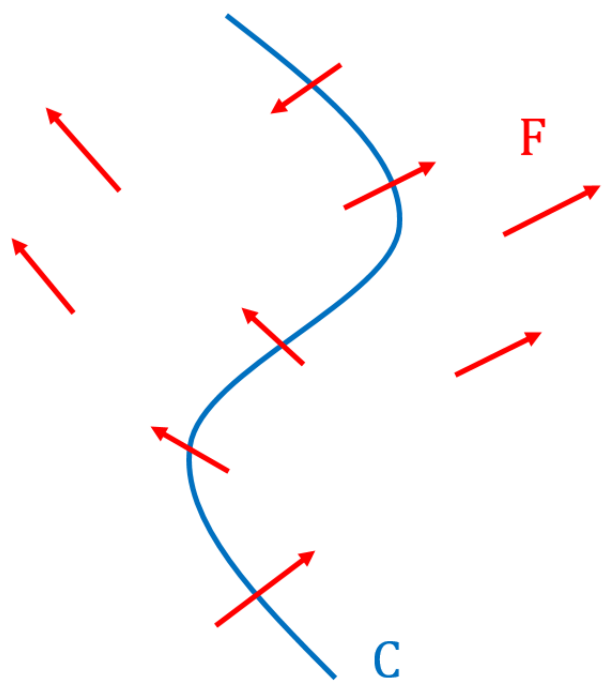
LECTURE 34: LINE INTEGRALS (II) + FTC (I)

1. Line Integral of a Vector Field

Video: Line Integral of a Vector Field

Goal: Given a vector field F and a curve C, want to sum up/ integrate the values of F along C

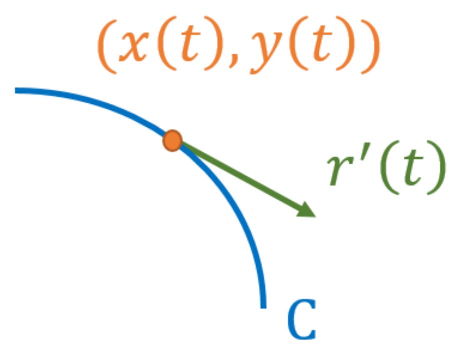


(Think of collecting all the arrows as you walk along C)

Date: Monday, November 15, 2021.

Notation:

$$r(t) = (x(t), y(t)) \text{ (The Curve)}$$
$$r'(t) = \langle x'(t), y'(t) \rangle \text{ (Tangent Vector)}$$



Definition: (Line Integral of F over C)

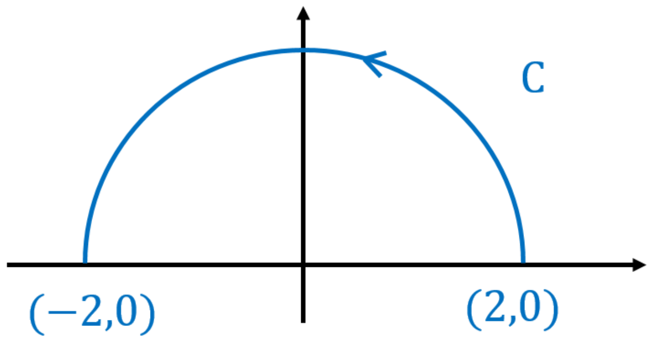
$$\int_C F \cdot dr = \int_a^b F \cdot \frac{dr}{dt} dt = \int_a^b F(r(t)) \cdot r'(t) dt$$

Example 1:

Calculate $\int_C F \cdot dr$, $F(x, y) = \langle x^2, -xy \rangle$

C : Half Circle from $(2, 0)$ to $(-2, 0)$ with $y \geq 0$, counterclockwise

STEP 1: Picture



STEP 2: Parametrize

$$\begin{cases} x(t) = 2\cos(t) \\ y(t) = 2\sin(t) \\ 0 \leq t \leq \pi \end{cases}$$

So $r(t) = (2\cos(t), 2\sin(t))$

STEP 3: Integrate

$$\begin{aligned}
&\int_C F \cdot dr \\
&= \int_0^\pi F \cdot \frac{dr}{dt} dt \\
&= \int_0^\pi F(r(t)) \cdot r'(t) dt \\
&= \int_0^\pi \left\langle (x(t))^2, -x(t)y(t) \right\rangle \cdot \langle x'(t), y'(t) \rangle \, dt \\
&= \int_0^\pi \left\langle 4\cos^2(t), -2\cos(t) 2\sin(t) \right\rangle \cdot \langle -2\sin(t), 2\cos(t) \rangle \, dt \\
&= \int_0^\pi -8\cos^2(t)\sin(t) - 8\cos(t)\sin(t)\cos(t) dt
\end{aligned}$$

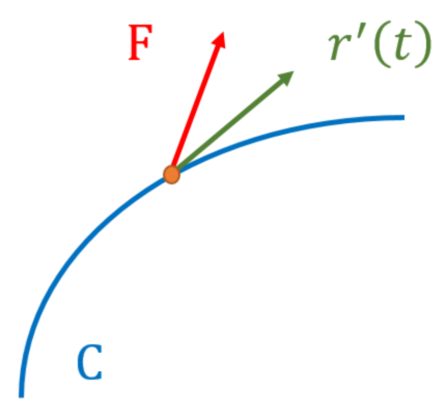
$$
\begin{aligned}
&= \int_0^{\pi} -16\cos^2(t)\sin(t)dt \\
&= \left[\frac{16}{3}\cos^3(t)\right]_0^{\pi} \\
&= -\frac{16}{3} - \frac{16}{3} \\
&= -\frac{32}{3}
\end{aligned}
$$

Note: If C were in the clockwise direction, then the answer would be $-\left(-\frac{32}{3}\right) = \frac{32}{3}$.

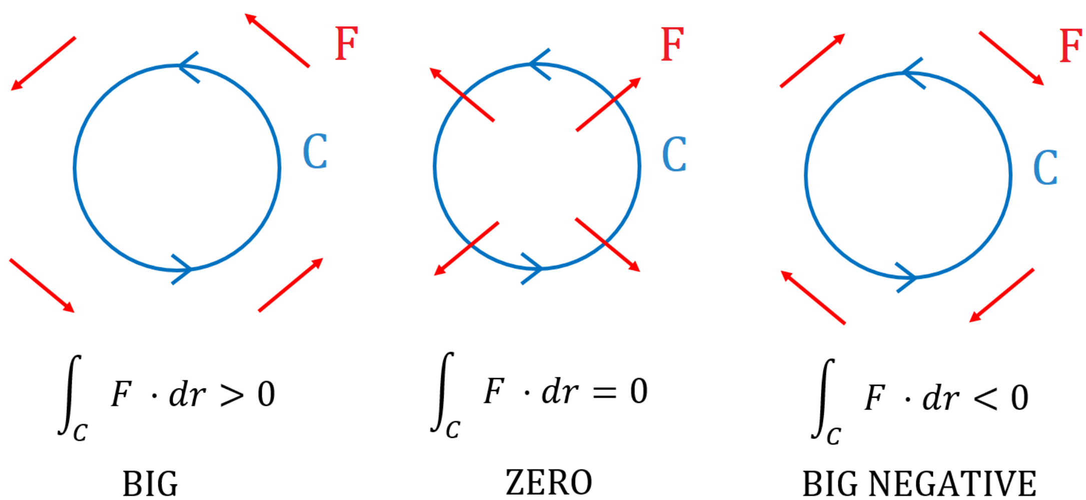
Applications/Intuition:

(1) If F = Force, then $\int_C F \cdot dr$ = Work done by F on C

(2) $F \cdot r'(t)$ is a number which measures how close F is to C, and $\int_C F \cdot dr = \int F \cdot r'(t)$ just sums up those numbers



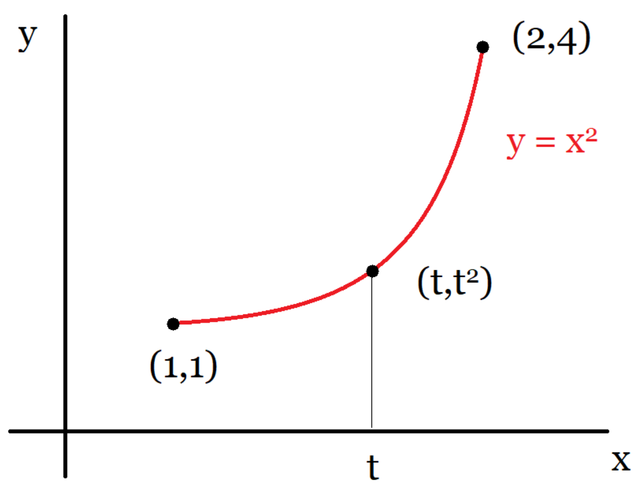
(3) **3 Scenarios:** In each scenario, think of running on a track C and F is wind blowing either with or against you:



Example 2: (more practice)

Find the work done by the force $F = \langle x\sin(y), y \rangle$ on the particle that moves along the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$

STEP 1: Picture



STEP 2: Parametrization

$$\begin{cases} x(t) = t \\ y(t) = t^2 \\ \quad (1 \leq t \leq 2) \end{cases}$$

STEP 3: Integrate

$$\begin{aligned}
&\int_C F \cdot dr \\
=&\int_1^2 \langle x(t)\sin(y(t)), y(t)\rangle \cdot \langle x'(t), y'(t)\rangle \, dt \\
=&\int_1^2 \left\langle t\sin\left(t^2\right), t^2\right\rangle \cdot \langle 1, 2t\rangle \, dt \\
=&\int_1^2 t\sin\left(t^2\right) + 2t^3 dt \\
=&\left[-\frac{1}{2}\cos\left(t^2\right) + \frac{1}{2}t^4\right]_1^2 \\
=&\frac{1}{2}\left(-\cos(4) + \cos(1)\right) + \frac{1}{2}(16-1) \\
=&\frac{1}{2}\left(\cos(1) - \cos(4) + 15\right)
\end{aligned}$$

2. Connecting the two

So far we talked about two different topics: Line Integrals of a function and line integrals of vector fields. It turns out they are both the same!

Example 3:

Consider $\int_C -ydx + xdy$

$$
\begin{aligned}
&\int_C -ydx + xdy \ \text{ (that shadow thing, from last time)} \\
&\int_a^b -y(t)x'(t) + x(t)y'(t)dt \\
=&\int_a^b \langle -y(t), x(t)\rangle \cdot \langle x'(t), y'(t)\rangle \, dt \\
=&\int_a^b F(r(t)) \cdot r'(t) \ \ F(x,y) = \langle -y, x\rangle \\
=&\int_C F \cdot dr \ \text{ (line integral of vector field)}
\end{aligned}
$$

So both topics are just two different sides of the same coin!

Take-Away:

If P and Q are functions, then

$$\int_C Pdx + Qdy = \int_C F \cdot dr \quad \text{where } F = \langle P, Q\rangle$$

3. FTC for line integrals (Section 16.3)

We are now ready for the first of four Fundamental Theorems of Calculus (FTC) in this course: The FTC for Line Integrals!

Recall: (FTC)

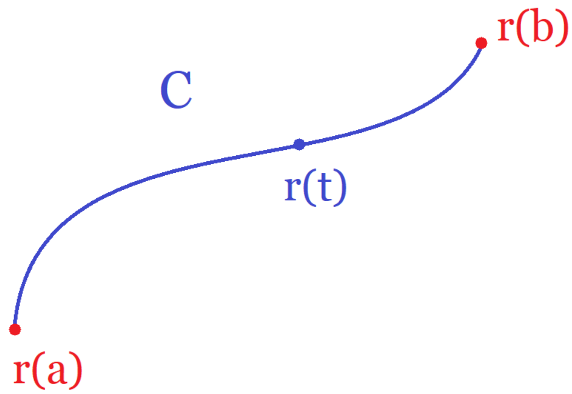
$$\int_a^b f'(x)dx = f(b) - f(a) = \ f(\text{end}) - f(\text{start})$$

Here it's the same thing, except we replace f' by ∇f and the integral by a line integral (the proof is in the optional appendix)

FTC for Line Integrals

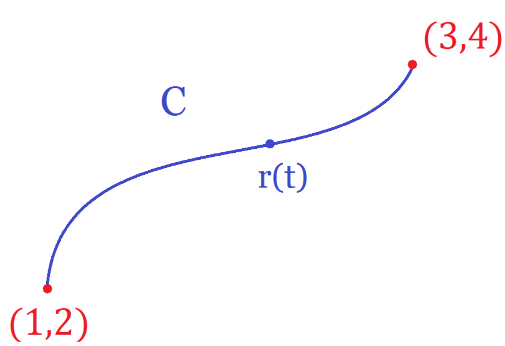
$$\int_C \nabla f \cdot dr = f(\text{end}) - f(\text{start}) = f(r(b)) - f(r(a))$$

(This says: Integral of a derivative is $f(b) - f(a)$)



Example 4:

Find $\int_C F \cdot dr$, where $F(x,y) = \langle xy^2, x^2y \rangle$ and C is any curve from $(1,2)$ to $(3,4)$



Can show: $F = \nabla f$, where $f(x,y) = \frac{1}{2}x^2y^2$, then:

$$\begin{aligned}\int_C F \cdot dr &= \int_C \nabla f \cdot dr \\ &= f(\text{end}) - f(\text{start}) \\ &= f(3,4) - f(1,2) \\ &= \frac{1}{2}(3)^2(4)^2 - \frac{1}{2}(1)^2(2)^2 \\ &= 70\end{aligned}$$

Take-Away

If F is conservative, $F = \nabla f$, then $\int_C F \cdot dr$ is easy to evaluate!

(This precisely answers the question from 16.1 as to why conservative vector fields are so nice!)

4. Conservative Vector Fields

Problem: How to determine if F is conservative?

It turns out that there is a really easy test for that!

⚠ This trick only works in 2 dimensions! (will find a 3D analog later)

2 dimensions: Suppose

$$\begin{aligned}F &= \nabla f \\ \langle P, Q \rangle &= \langle f_x, f_y \rangle \\ P = f_x &\quad Q = f_y\end{aligned}$$

Recall: (Clairaut)

$$
\begin{aligned}
f_{xy} &= f_{yx} \\
(f_x)_y &= (f_y)_x \\
P_y &= Q_x
\end{aligned}
$$

Fact:

If $F = \langle P, Q \rangle$ is conservative, then $P_y = Q_x$

Mnemonic: Peyam = Quixotic

Example 5:

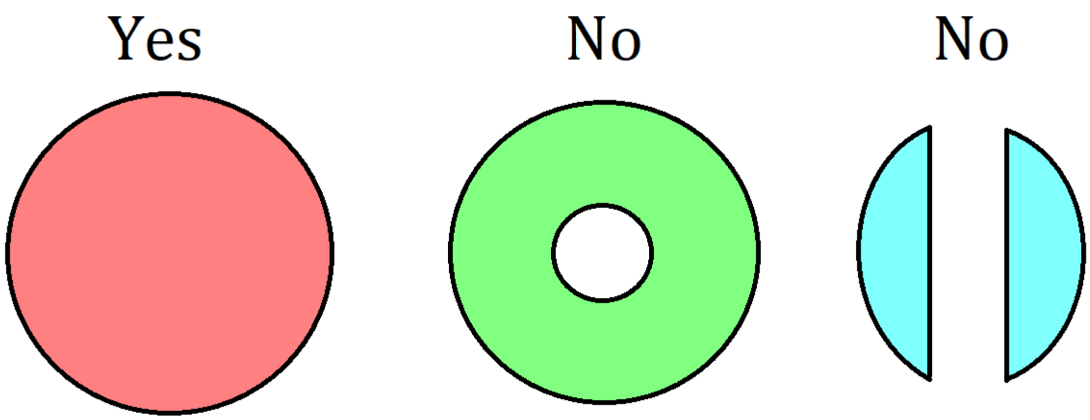
Is $F = \langle -y, x \rangle$ (rotation field) conservative?

$$
\begin{aligned}
P = -y,\ Q = x \\
P_y = -1,\ Q_x = 1 \\
P_y \neq Q_x \\
\text{No}
\end{aligned}
$$

So Conservative $\Rightarrow P_y = Q_x$.

Question: Does $P_y = Q_x \Rightarrow F$ conservative? “Yes”

(Yes if the domain of F has no holes, no otherwise)



Important Fact: (if no holes)

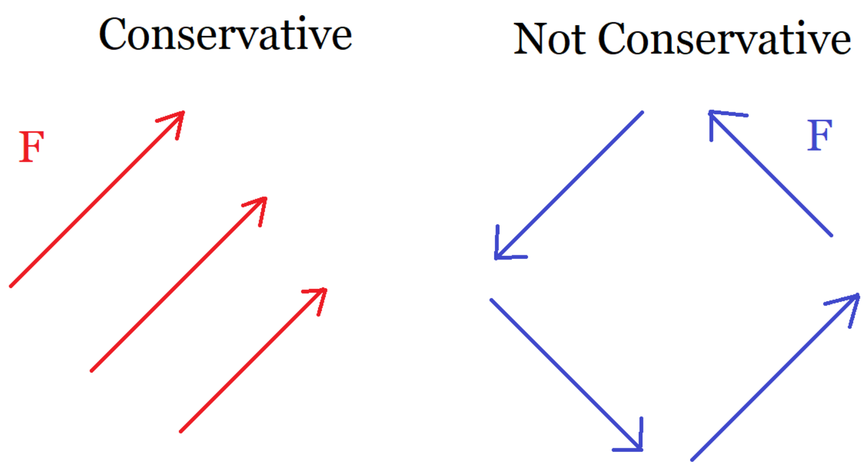
$$F \text{ conservative} \Leftrightarrow P_y = Q_x$$

Example 6:

Is $F = \left\langle 3 + 2xy, x^2 - 3y^2 \right\rangle$ conservative?

$$\begin{aligned} P_y &= (3 + 2xy)_y = 2x \\ Q_x &= (x^2 - 3y^2)_x = 2x \\ P_y &= Q_x \\ &\text{Yes} \end{aligned}$$

Intuitively: Conservative means "Doesn't Rotate" and not conservative means "Rotates"



5. Finding antiderivatives

Now suppose F is conservative, how to find an antiderivative of F?

Example 7:

Let $F = \left\langle 3 + 2xy, x^2 - 3y^2 \right\rangle$, find f such that $F = \nabla f$

STEP 1: Check $P_y = Q_x$ ✓(see previous example)

STEP 2: $F = \nabla f \Rightarrow \left\langle 3 + 2xy, x^2 - 3y^2 \right\rangle = \langle f_x, f_y \rangle$, hence:

$$f_x(x, y) = 3 + 2xy \Rightarrow f(x, y) = \int 3 + 2xy \, dx = 3x + x^2 y + \text{ JUNK}$$

This is saying that f has $3x$ and $x^2 y$ in it, with possibly other terms

$$f_y(x, y) = x^2 - 3y^2 \Rightarrow f(x, y) = \int x^2 - 3y^2 dy = x^2 y - y^3 + \text{ JUNK}$$

Now collect all the terms (x^2y appears twice, don't double-count it)

STEP 3:

$$f(x,y) = x^2y + 3x - y^3$$

(There might be other possibilities, but just need *one* antiderivative)

Example 8: (more practice)

Find f such that

$$F(x,y,z) = \left\langle y^2, 2xy + e^{3z}, 3ye^{3z} \right\rangle = \nabla f = \langle f_x, f_y, f_z \rangle$$

STEP 1: Check F conservative. See 16.5 ✓

STEP 2:

$$f_x(x,y,z) = y^2 \Rightarrow f(x,y,z) = \int y^2 dx = xy^2 + \text{ JUNK}$$

$$f_y(x,y,z) = 2xy + e^{3z} \Rightarrow f(x,y,z) = \int 2xy + e^{3z} dy = xy^2 + ye^{3z} + \text{ JUNK}$$

$$f_z(x,y,z) = 3ye^{3z} \Rightarrow f(x,y,z) = \int 3ye^{3z} dz = 3y\left(\frac{e^{3z}}{3}\right) = ye^{3z} + \text{ JUNK}$$

STEP 3: Hence $f(x,y,z) = xy^2 + ye^{3z}$

6. Appendix: Proof of FTC

$$\text{Consider: } \int_a^b \frac{d}{dt} f(r(t))dt$$

On the one hand, this equals

$$\int_a^b \frac{d}{dt} f(r(t))dt = f(r(b)) - f(r(a))$$

On the other hand, by the Chen Lu (Chain Rule):

$$\begin{aligned}
\frac{d}{dt} f(r(t)) =& \frac{d}{dt} f(x(t), y(t)) \\
=& \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\
=& (f_x)(x'(t)) + (f_y)(y'(t)) \\
=& \langle f_x, f_y \rangle \cdot \langle x'(t), y'(t) \rangle \\
=& \nabla f(x(t), y(t)) \cdot r'(t) \\
=& \nabla f(r(t)) \cdot r'(t)
\end{aligned}$$

$$\text{Hence: } \int_a^b \frac{d}{dt} f(r(t))dt = \int_a^b \nabla f(r(t)) \cdot r'(t) = \int_C \nabla f \cdot dr$$

Combining the two, we get:

$$\int_C \nabla f \cdot dr = \int_a^b \frac{d}{dt} f(r(t))dt = f(r(b)) - f(r(a))$$