LECTURE 35: FTC FOR LINE INTEGRALS (II)

1. Putting it together

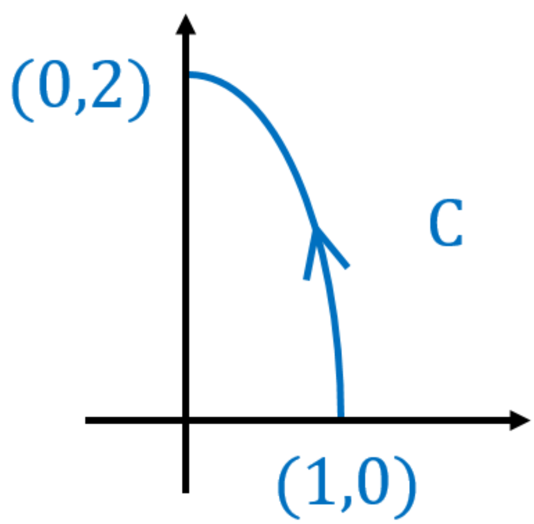
Video: FTC for Line Integrals

Example 1:

Calculate $\int_C F \cdot dr$ where $F(x,y) = \left\langle x^2y^3, x^3y^2 \right\rangle$ and C is the curve

$$\begin{cases} x(t) = \cos(t) \\ y(t) = 2\sin(t) \\ 0 \leq t \leq \frac{\pi}{2} \end{cases}$$

STEP 1: Picture:



Date: Wednesday, November 17, 2021.

Could calculate the integral directly, but it becomes way harder (sometimes even impossible) to integrate

STEP 2: Check:

$$P_y = \left(x^2y^3\right)_y = 3x^2y^2$$
$$Q_x = \left(x^3y^2\right)_x = 3x^2y^2$$

STEP 3: Find f

$$F = \nabla f \Rightarrow \left\langle x^2y^3, x^3y^2\right\rangle = \langle f_x, f_y\rangle$$

$$f_x(x,y) = x^2y^3 \Rightarrow f(x,y) = \int x^2y^3dx = \frac{1}{3}x^3y^3 + \text{ JUNK}$$

$$f_y(x,y) = x^3y^2 \Rightarrow f(x,y) = \int x^3y^2dy = \frac{1}{3}x^3y^3 + \text{ JUNK}$$

$$f(x,y) = \frac{1}{3}x^3y^3$$

STEP 4: Integrate

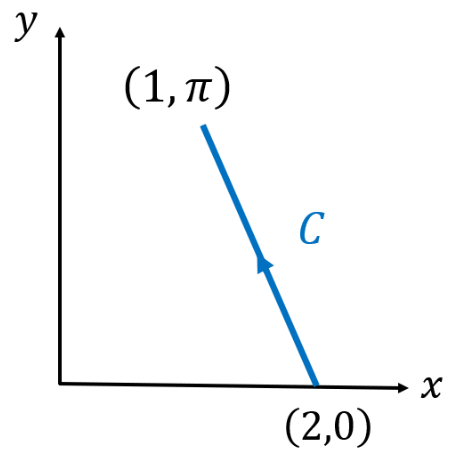
$$\begin{aligned}\int_C F \cdot dr &= \int_C \nabla f \cdot dr \\ &= f(\text{end}) - f(\text{start}) \\ &= f(0,2) - f(1,0) \\ &= \frac{1}{3}(0)^3(2)^3 - \frac{1}{3}(1)^3(0)^3 \\ &= 0\end{aligned}$$

Video: FTC Example

Example 2:

Calculate $\int_C F \cdot dr$ where $F(x,y) = \langle \sin(y), x\cos(y) - \sin(y)\rangle$ and C is the line from $(2,0)$ to $(1,\pi)$

STEP 1: Picture:



STEP 2: Conservative:

$$\begin{aligned} P_y =& (\sin(y))_y = \cos(y) \\ Q_x =& (x\cos(y) - \sin(y))_x = \cos(y) \\ P_y =& Q_x \checkmark \end{aligned}$$

STEP 3: Antiderivative:

$$F = \nabla f \Rightarrow \langle \sin(y), x\cos(y) - \sin(y)\rangle = \langle f_x, f_y\rangle$$

$$\begin{aligned} f_x =& \sin(y) \Rightarrow f(x,y) = \int \sin(y) dx = x\sin(y) + \text{ JUNK} \\ f_y =& x\cos(y) - \sin(y) \\ &\Rightarrow f(x,y) = \int x\cos(y) - \sin(y) dy = x\sin(y) + \cos(y) + \text{ JUNK} \end{aligned}$$

$$f(x,y) = x\sin(y) + \cos(y)$$

STEP 4:

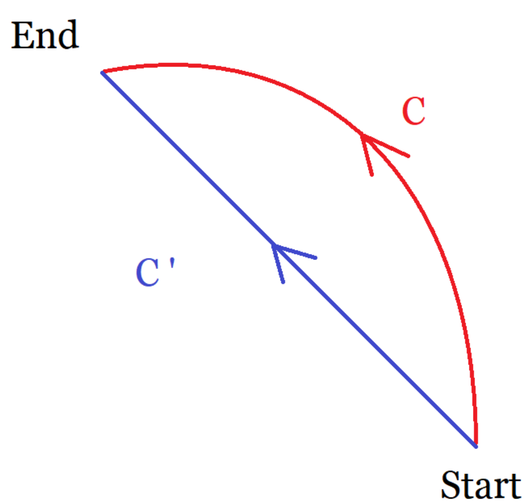
$$\begin{aligned}\int_C F \cdot dr =& \int_C \nabla f \cdot dr \\ =& f(1,\pi) - f(2,0) \\ =& 1\sin(\pi) + \cos(\pi) - 2\sin(0) - \cos(0) \\ =& -1-1 \\ =& -2\end{aligned}$$

Note: There are more examples at the end (if you need more practice)

2. Path (in)-dependence

Video: Path Independence

Recall: In general, $\int_C F \cdot dr$ depends on the path C



$\int_C F \cdot dr \neq \int_{C'} F \cdot dr$, even if C and C' have the same start/endpoints.

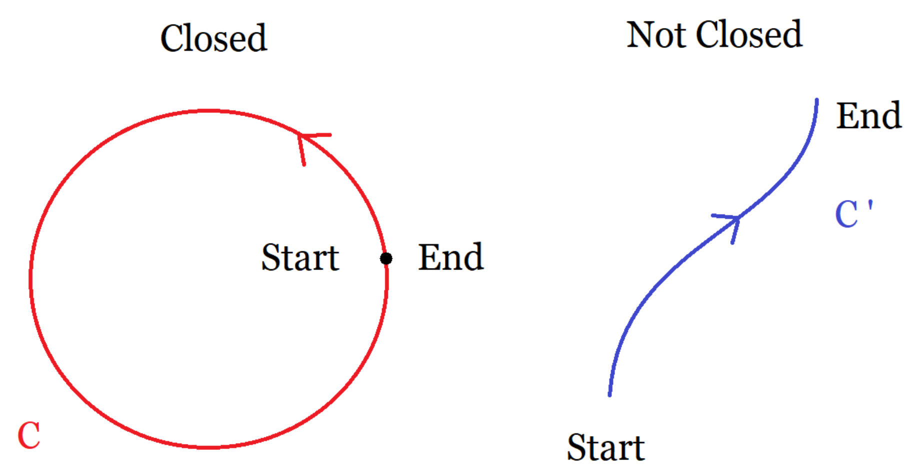
Question:

When is $\int_C F \cdot dr$ **in**dependent of the path?

For this, we'll need a quick definition:

Definition:

C is **closed** if Start = End



The following is an easy test for path independence:

Neat Fact:

$$\int_C F \cdot dr \text{ is independent of path } \Leftrightarrow \int_C F \cdot dr = 0 \text{ for every } \underline{\text{closed}}\ C$$

In other words, to check for independence, just don't need to check this for all paths, just for closed ones.

(For a proof of the neat fact, check out the Appendix at the end)

From this, we get an even easier and more important fact:

Important Fact:

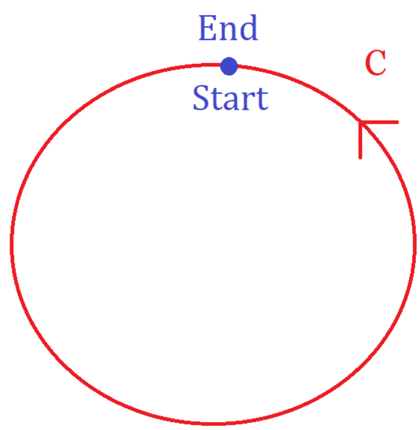
$\int_C F \cdot dr$ is independent of path $\Leftrightarrow F$ is conservative $(F = \nabla f)$

Which explains **YET AGAIN** why conservative vector fields are important!

Proof:

$(\Rightarrow)$ Skip (but it explicitly constructs f)

$(\Leftarrow)$ Let's use the neat fact! Suppose $F = \nabla f$ and assume C is closed, then:



$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(\text{end}) - f(\text{start}) = 0 \text{ (since } C \text{ is closed)}$$

Since $\int_C F \cdot dr = 0$ for all closed C, we're done by the Neat Fact $\square$

Remark: If C is closed, $\int_C F \cdot dr$ is sometimes called the **circulation** of F around C and measures how many times F loops around C. For conservative F, we have $\int_C F \cdot dr = 0$, so conservative F are *irrotational* (= don't rotate)

Summary:

Conservative vector fields $F = \nabla f$ are nice because:

(1) $\int_C F \cdot dr$ is easy to calculate (by FTC)

(2) $\int_C F \cdot dr$ is independent of path

(3) F is irrotational

3. More Examples and Pitfalls

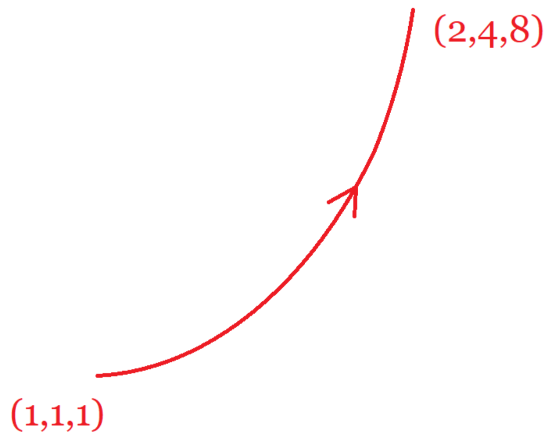
Video: FTC 3 Dimensions

Example 3:

Calculate $\int_C F \cdot dr$, where $F(x, y, z) = \langle yze^{xz}, e^{xz}, xye^{xz} \rangle$ and C is the curve:

$$\begin{cases} x(t) = t \\ y(t) = t^2 \\ z(t) = t^3 \\ 1 \leq t \leq 2 \end{cases}$$

STEP 1: Picture:



STEP 2: Conservative: See Section 16.5

STEP 3: Antiderivative:

$$F = \nabla f \Rightarrow \langle yze^{xz}, e^{xz}, xye^{xz} \rangle = \langle f_x, f_y, f_z \rangle$$

$$\begin{aligned} f_x =& yze^{xz} \Rightarrow f(x,y,z) = \int yze^{xz} dx = yz\left(\frac{e^{xz}}{z}\right) + \text{ JUNK } = ye^{xz} + \text{ JUNK} \\ f_y =& e^{xz} \Rightarrow f(x,y,z) = \int e^{xz} dy = ye^{xz} + \text{ JUNK} \\ f_z =& xye^{xz} \Rightarrow f(x,y,z) = \int xye^{xz} dz = xy\left(\frac{e^{xz}}{x}\right) + \text{ JUNK } = ye^{xz} + \text{ JUNK} \end{aligned}$$

$$f(x,y,z) = ye^{xz}$$

STEP 4: Integrate

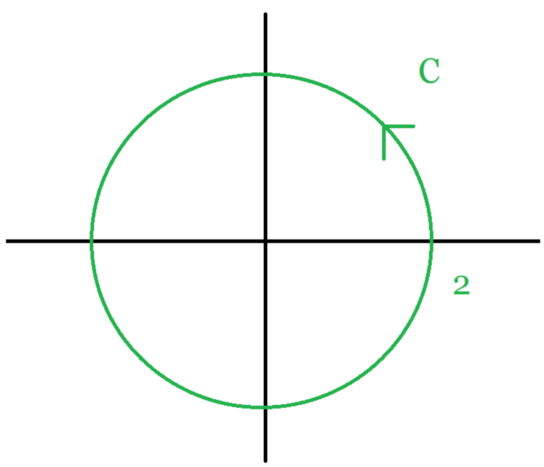
$$\begin{aligned}&\int_C F\cdot dr\\=&\int_C \nabla f\cdot dr\\=&f(2,4,8)-f(1,1,1)\\=&4e^{(2)(8)}-1e^{(1)(1)}\\=&4e^{16}-e\end{aligned}$$

Video: FTC Pitfalls

Example 4:

$\int_C F\cdot dr$, $F(x,y)=\langle 2y,3x\rangle$, C : Circle centered at $(0,0)$ of radius 2 (counterclockwise)

STEP 1: Picture:



STEP 2: Conservative:

$$\begin{cases} P_y = 2 \\ Q_x = 3 \end{cases} \Rightarrow P_y \neq Q_x \Rightarrow \text{ NO}$$

RUH-OH!!! Well, in that case you have to get your hands dirty and calculate the integral directly.

STEP 3: Parametrize:

$$\begin{cases} x(t) = & 2\cos(t) \\ y(t) = & 2\sin(t) \\ 0 \leq t \leq 2\pi \end{cases}$$

STEP 4: Integrate:

$$\begin{aligned} & \int_C F \cdot dr \\ = & \int_0^{2\pi} F(r(t)) \cdot r'(t) dt \\ = & \int_0^{2\pi} \langle 2(2\sin(t)), 3(2\cos(t)) \rangle \cdot \langle -2\sin(t), 2\cos(t) \rangle \, dt \\ = & \int_0^{2\pi} -8\sin^2(t) + 12\cos^2(t) dt \end{aligned}$$

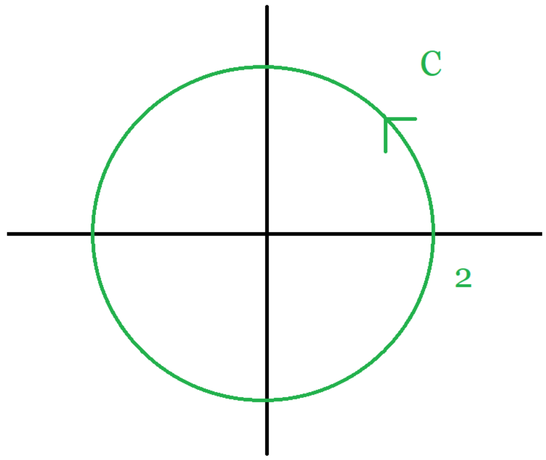
$$
\begin{aligned}
&= \int_0^{2\pi} -8\sin^2(t) + 12\left(1 - \sin^2(t)\right) dt \\
&= \int_0^{2\pi} -8\sin^2(t) + 12 - 12\sin^2(t) dt \\
&= \int_0^{2\pi} 12 - 20\sin^2(t) dt \\
&= 12(2\pi) - 20\int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos(2t) dt \\
&= 24\pi - 20\left[\frac{t}{2} - \frac{1}{4}\cos(2t)\right]_0^{2\pi} \\
&= 24\pi - 20\left(\pi - 0 - \frac{1}{4}\cos(2\pi) + \frac{1}{4}\cos(0)\right) \\
&= 24\pi - 20\pi \\
&= 4\pi
\end{aligned}
$$

Notice: Even though C is closed, we have $\int_C F \cdot dr \neq 0$, yet another argument why F is not conservative.

Example 5: (extra practice)

Same, but $F(x, y) = \langle 2xy, x^2 \rangle$

STEP 1: Picture:



STEP 2: Conservative:

$$\begin{aligned} P_y &= 2x \\ Q_x &= 2x \\ P_y &= Q_x \checkmark \end{aligned}$$

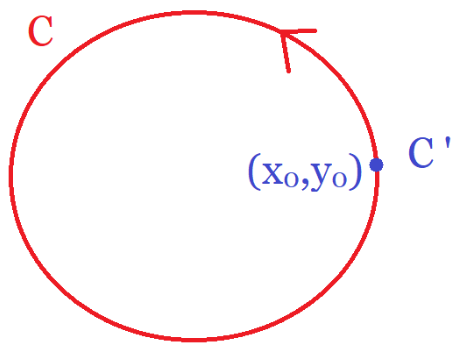
But since C is closed, we automatically get $\int_C F \cdot dr = 0$.

4. Appendix: Proof of Neat Fact

Neat Fact:

$$\int_C F \cdot dr \text{ is independent of path } \Leftrightarrow \int_C F \cdot dr = 0 \text{ for every } \underline{\text{closed}}\ C$$

($\Rightarrow$) Suppose the integral is independent of path and let C be any closed curve. Pick any point (x_0, y_0) on C and let C' be the path parametrized by $r(t) = (x_0, y_0)$ (so C' literally does nothing)

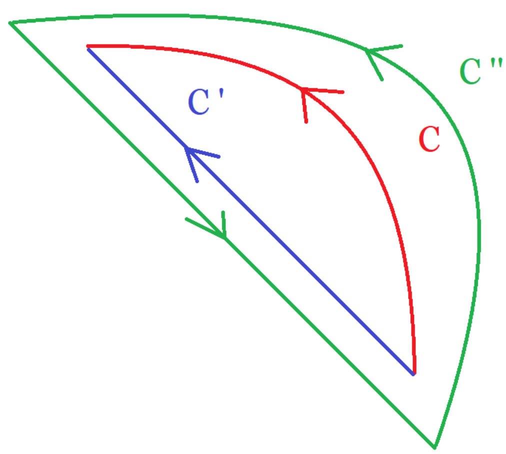


Then $\int_{C'} F \cdot dr = 0$, since $r(t) = (x_0, y_0)$ and $r'(t) = \langle 0, 0 \rangle$. But since the integral is independent of path, we get

$$\int_C F \cdot dr = \int_{C'} F \cdot dr = 0$$

($\Leftarrow$) Suppose C and C' are two curves with the same start and end-points, we want to show $\int_C F \cdot dr = \int_{C'} F \cdot dr$

Let C'' be C followed by C' (but in the opposite direction). So C'' is the loop formed by C and C'



Then, since C'' is closed, by assumption we get:

$$\begin{aligned}
\int_{C''} F \cdot dr &= 0 \\
\int_C F \cdot dr + \int_{-C'} F \cdot dr &= 0 \quad -C' \text{ is } C' \text{ but in the other direction} \\
\int_C F \cdot dr - \int_{C'} F \cdot dr &= 0 \\
\int_C F \cdot dr = \int_{C'} F \cdot dr
\end{aligned}$$