LECTURE 36: GREEN'S THEOREM

Welcome to the second FTC for vector fields! They say the grass is greener on the other side, but I say that the grass is Green's Theorem-er on the other side, because today is all about Green's Theorem!

1. Motivation

⚠ This only works for 2 dimensions! There are analogs in 3 dimensions, which we'll cover later, in sections 16.8 and 16.9

What makes Green's Theorem so useful is that it also works even if F is not conservative

$$\textbf{Motivation:} \quad \text{FTC: } F = \int F' \Rightarrow \int F = \int \int F'$$

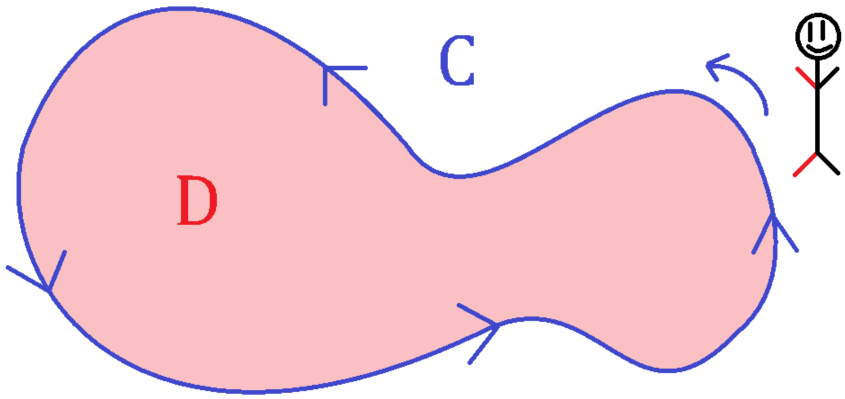
The left hand side is just $\int_C F \cdot dr$ and the right-hand-side becomes Quixotic Peyams:

Green's Theorem:

If $F = \langle P, Q \rangle$, and C is a closed curve with inside region D, then

$$\int_C F \cdot dr = \int \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx dy$$

Date: Friday, November 19, 2021.



So instead of calculating a hard line integral (left), you calculate an easier double integral (right)

Mnemonic: QuiXotic PeYams, or:

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} = \frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Note: Orientation matters! Make sure that if you walk along the curve C, then your region D is to your left (WALK Left), otherwise it's minus your answer. This is called positive orientation.

2. Examples

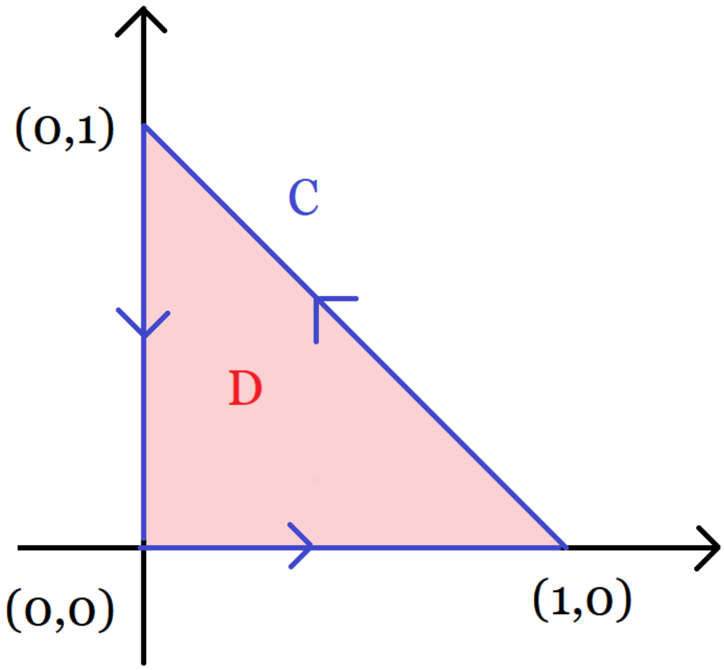
Video 1: Green's Theorem Example 1
Video 2: Green's Theorem Example 2

Green's theorem is useful for calculating line integrals.

Example 1:

Calculate $\int_C F \cdot dr$, where: $F(x,y) = \langle x^4, xy \rangle$ and C is the Triangle connecting $(0,0), (1,0), (0,1)$, counterclockwise

STEP 1: Picture:

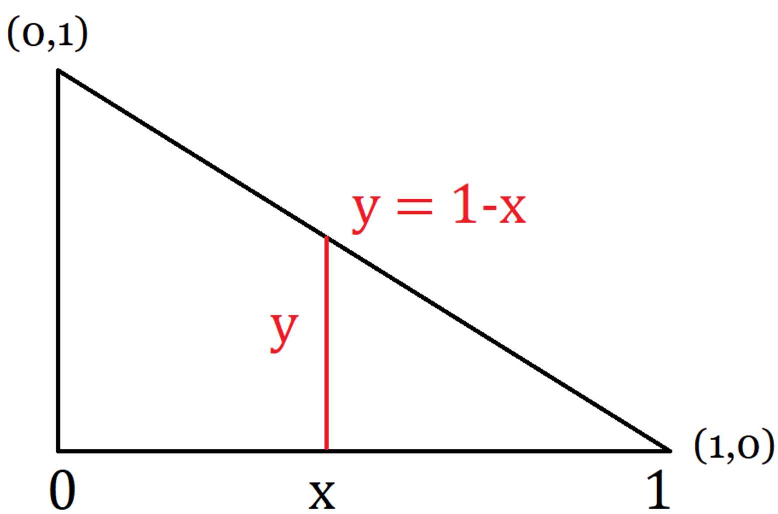


Notice: It's a **PAIN** to do the line integral directly! Not only is F complicated, but you also need to split the line integral up into 3 pieces!

STEP 2: Integrate: (D is to your left, so the orientation checks out)

$$\begin{aligned}
&\int_C F \cdot dr \\
&= \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy \\
&= \int\int_D (xy)_x - \left(x^4\right)_y dxdy \\
&= \int\int_D y dxdy
\end{aligned}$$

STEP 3:



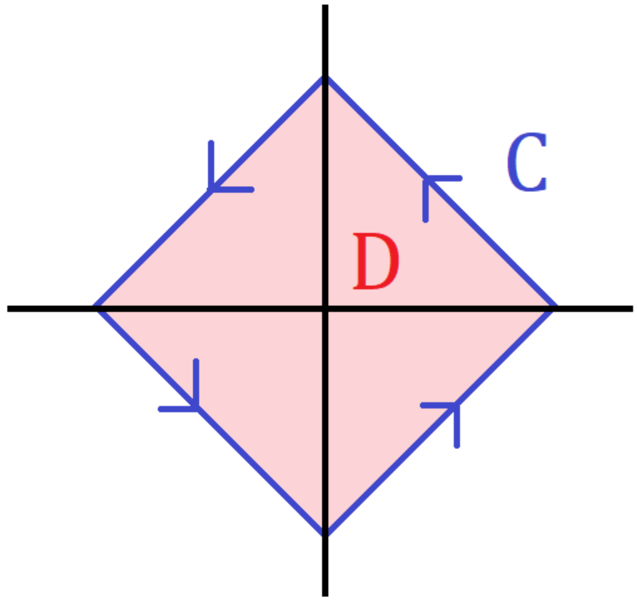
$$\begin{cases} 0 \leq y \leq 1 - x \\ 0 \leq x \leq 1 \end{cases}$$

$$\begin{aligned}
&\int\int_D y dx dy \\
=&\int_0^1 \int_0^{1-x} y dy dx \\
=&\int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{y=1-x} \\
=&\int_0^1 \frac{(1-x)^2}{2} dx \\
=&\left[-\frac{1}{6}(1-x)^3\right]_0^1 \\
=&\frac{1}{6}
\end{aligned}$$

Example 2:

Calculate $\int_C F \cdot dr$, where: $F(x,y) = \left\langle 3y - e^{\sin(x)}, 7x + \sqrt{y^4+1} \right\rangle$ and C is the square with vertices $(\pm 1, 0), (0, \pm 1)$, counterclockwise

STEP 1: Picture:

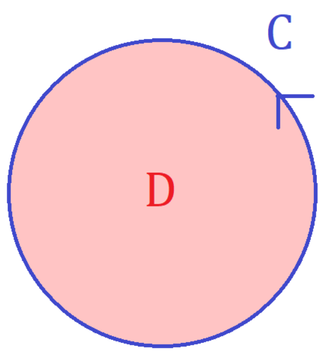


STEP 2: Integrate:

$$
\begin{aligned}
&\int_C F \cdot dr \\
=&\int\int_D \frac{\partial}{\partial x}\left(7x + \sqrt{y^4+1}\right) - \frac{\partial}{\partial y}\left(3y - e^{\sin(x)}\right) dxdy \\
=&\int\int_D 7 - 3 dxdy \\
=&\int\int_D 4 dxdy \\
=&4\int\int_D 1 dxdy \\
=&4 \text{ Area } (D) \\
=&4(\sqrt{2})^2 \\
=&8
\end{aligned}
$$

Remark: Last time, we showed that **if** $F = \langle P, Q \rangle$ conservative **then** $P_y = Q_x$ (by Clairaut)

Now **IF** $P_y = Q_x$ (and no holes), then for any closed curve C,



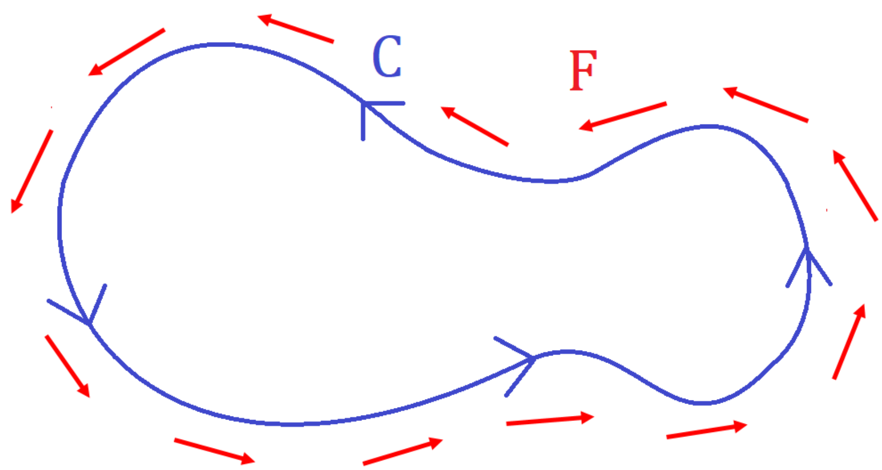
$$\int_C F \cdot dr = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

So F is conservative (by Neat Fact from last time)

Fact: (again)

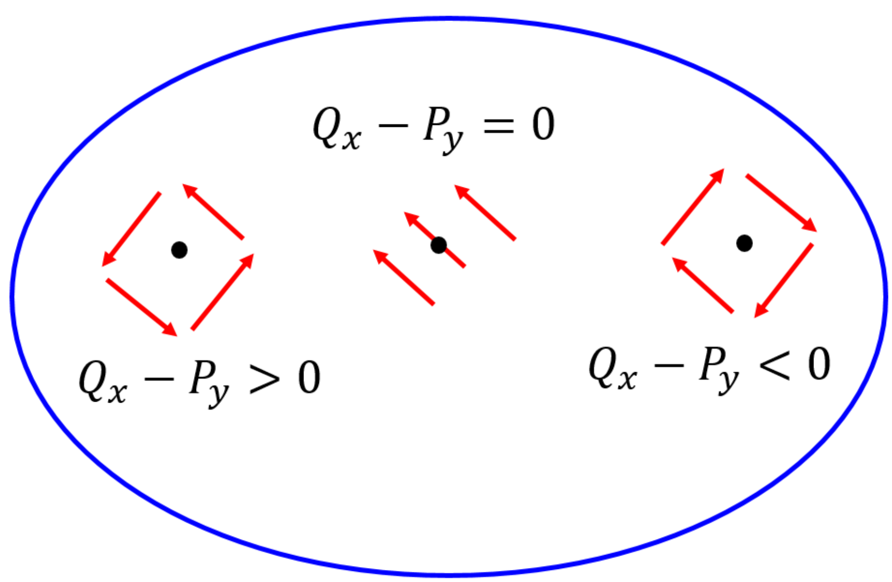
$$F \text{ conservative } \Leftrightarrow P_y = Q_x$$

3. Some Intuition



$\int_C F \cdot dr$ measures the circulation of F around C (think F = wind or water) which you can think of a **macroscopic rotation**

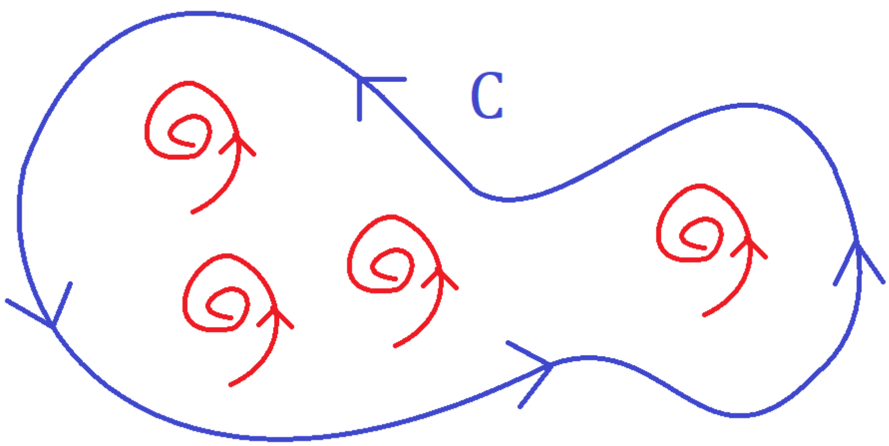
$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ measures the rotation of F around a point, which is a **microscopic rotation**



Green's Theorem says:

$$\underbrace{\int\int \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy}_{\text{Sum of Microscopic Rotations}} = \underbrace{\int_C F \cdot dr}_{\text{Macroscopic Rotation}}$$

Which kind of makes sense! Think of the microscopic rotations as mini-whirlpools (or hurricanes) in a bath-tub C



Green's Theorem says that if you add up all the whirlpools inside the bathtub, you get a gigantic whirlpool/circulation around C.

4. Area 51

What makes Green's theorem exciting is not only the fact that it simplifies integrals, but especially its applications! Here we do two really cool ones.

So far: We saw that Green's Theorem helps us simplify line integrals. Now you may ask: Is the opposite true? Could we use Green's theorem to simplify double integrals? Not really **except** for one special case:

Recall

$$\text{Area }(D) = \int\int_D 1\, dxdy$$

So **IF** $F = \langle P, Q \rangle$ is such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then:

$$\int_C F \cdot dr \stackrel{G}{=} \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy = \int\int_D 1 dxdy = \text{ Area }(D)$$

Many choices for P and Q such that $Q_x - P_y = 1$

(Examples: $P = 0, Q = x$ or $P = -y, Q = 0$)

"Best" choice: $P = -\frac{y}{2}, Q = \frac{x}{2}$, which gives:

$$F = \langle P, Q \rangle = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle = \frac{1}{2} \langle -y, x \rangle \to \frac{1}{2}(-ydx + xdy)$$

FACT (Memorize)

$$\text{Area }(D) = \frac{1}{2} \int_C xdy - ydx$$

Mnemonic: $\frac{1}{2} \begin{vmatrix} x & y \\ dx & dy \end{vmatrix} = \frac{1}{2}(xdy - ydx)$

5. OMG Example

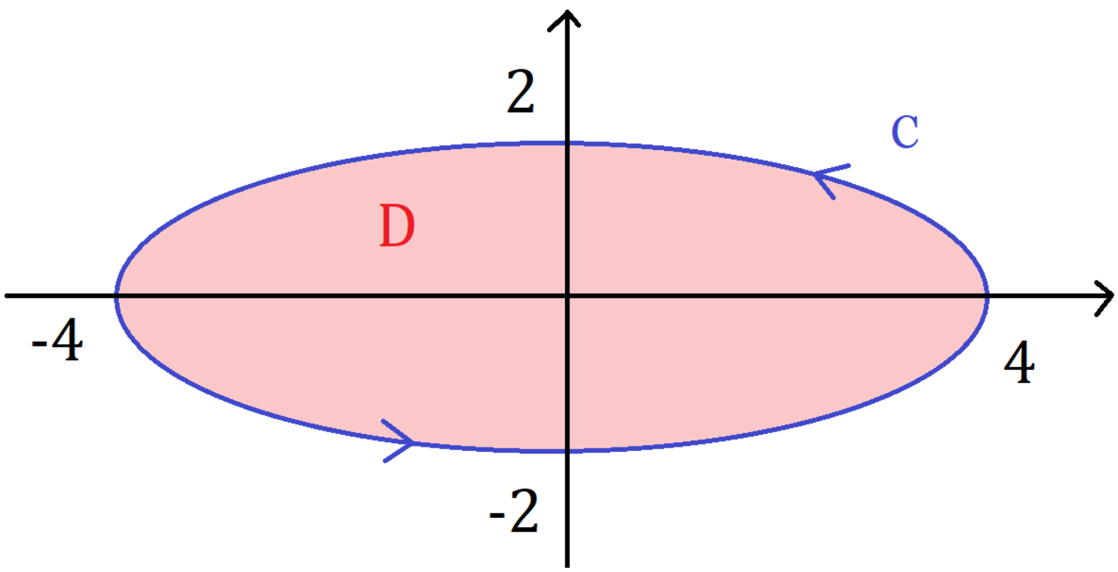
Video: Area of Ellipse

Example 3

Find the area enclosed by the ellipse

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

STEP 1: Picture:



STEP 2: Parametrize:

$$\begin{cases} x(t) = 4\cos(t) \\ y(t) = 2\sin(t) \\ 0 \leq t \leq 2\pi \end{cases}$$

STEP 3: Integrate

$$
\begin{aligned}
&\text{Area } (D)\\
=&\frac{1}{2}\int_C x\frac{dy}{dt} - y\frac{dx}{dt}dt\\
=&\frac{1}{2}\int_0^{2\pi} x(t)y'(t) - y(t)x'(t)dt\\
=&\frac{1}{2}\int_0^{2\pi} 4\cos(t)2\cos(t) - 2\sin(t)\left(-4\sin(t)\right)dt\\
=&\frac{1}{2}\int_0^{2\pi} \underbrace{8\cos^2(t) + 8\sin^2(t)}_{8}dt\\
=&\left(\frac{1}{2}\right)(8)(2\pi)\\
=&8\pi
\end{aligned}
$$

OMG, look how effortless this was!

6. OMGGG Example

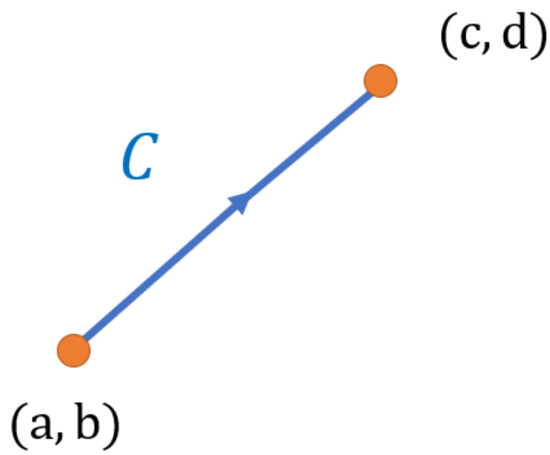
Video: Area of a Polygon

You might say "OMG Peyam, there's no way this could be even more exciting!!!" Wait for it... ☺

Example 4:

(a) (Prep Work:) Find $\int_C xdy - ydx$, C : Line connecting (a, b) to (c, d)

STEP 1: Picture:



STEP 2: Parametrize:

$$\begin{cases} x(t) = & (1-t)a + tc = \quad a + t(c-a) \\ y(t) = & (1-t)b + td = \quad c + t(d-b) \\ & 0 \leq t \leq 1 \end{cases}$$

STEP 3: Integrate

$$\begin{aligned} \int_C xdy - ydx &= \int_0^1 x(t)y'(t) - y(t)x'(t)dt \\ &= \int_0^1 \left(a + t(c-a)\right)(d-b) - \left(b + t(d-b)\right)(c-a)dt \\ &= \int_0^1 a\,(d-b) + \cancel{t\,(c-a)\,(d-b)} - b(c-a) - \cancel{t\,(d-b)\,(c-a)}dt \\ &= \int_0^1 ad - \cancel{ab} - bc + \cancel{ab}dt \\ &= ad - bc \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \end{aligned}$$

Therefore:

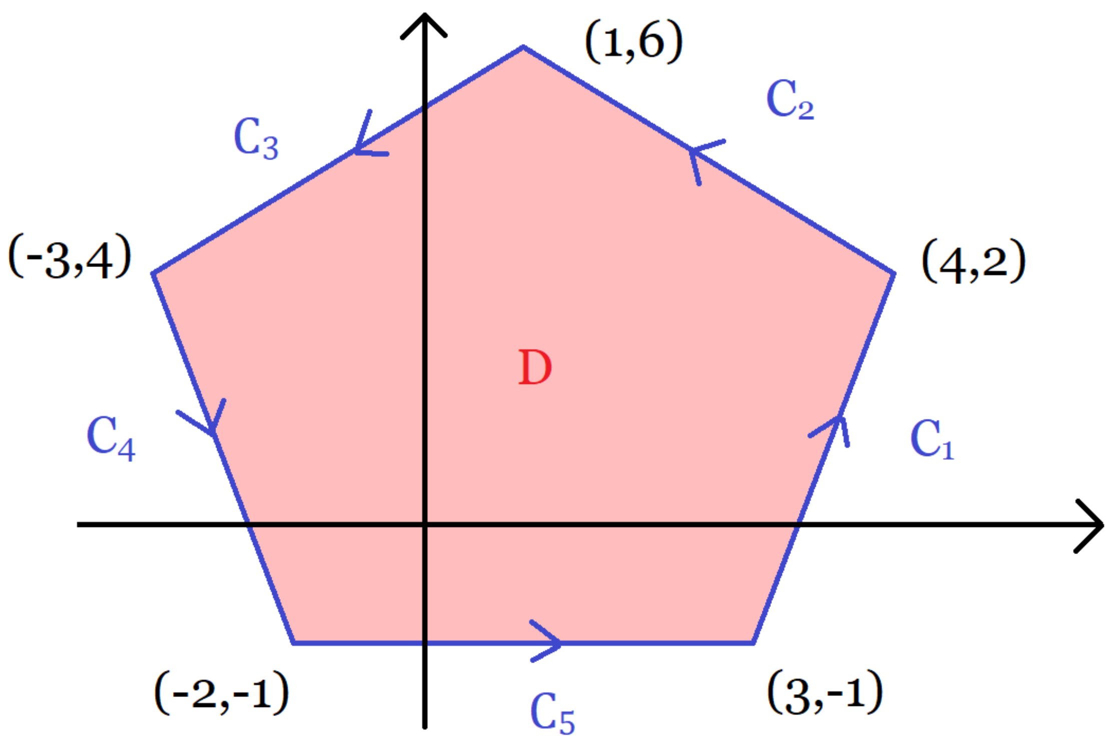
$$\int_C xdy - ydx = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

OMG Part

(b) Find the area of the pentagon with vertices $(3,-1),(4,2),(1,6),(-3,4),(-2,-1)$

(In fact, *any* polygon works)

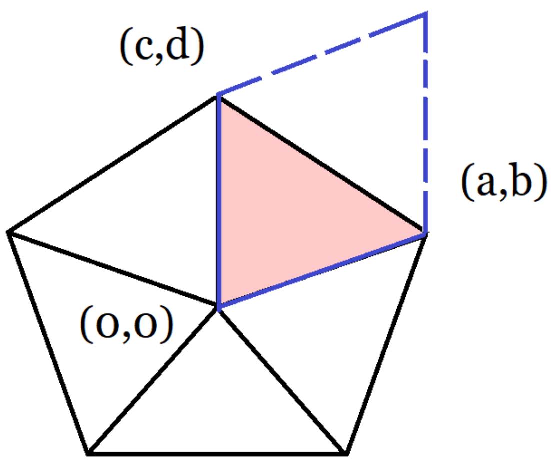
STEP 1: Picture:



STEP 2: Integrate:

$$\begin{aligned}
\text{Area } (D) =& \frac{1}{2}\int_C xdy - ydx \\
=& \frac{1}{2}\left(\int_{C_1} xdy - ydx + \int_{C_2} xdy - ydx + \cdots + \int_{C_5} xdy - ydx\right) \\
=& \frac{1}{2}\left(\begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ 1 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 6 \\ -3 & 4 \end{vmatrix} + \begin{vmatrix} -3 & 4 \\ -2 & -1 \end{vmatrix} + \begin{vmatrix} -2 & -1 \\ 3 & -1 \end{vmatrix}\right) \\
=& \frac{1}{2}(10 + 22 + 22 + 11 + 5) \\
=& 35 \text{ BOOM!!!}
\end{aligned}$$

Why does this work?



A pentagon (or any polygon) is the sum of triangles, which are half-parallelograms, so

$$\begin{aligned}\text{Area(Pentagon)} &= \text{Sum of Areas of Triangles} \\ &= \frac{1}{2}\left(\text{ Sum of areas of Parallelograms }\right) \\ &= \frac{1}{2}\left(\text{ Sum of } \begin{vmatrix} a & b \\ c & d \end{vmatrix}\right) \text{ (from Linear Algebra)}\end{aligned}$$

Note: You can even use this to calculate the volume of a polyhedron (3D polygon), which you can check out in this video (but it requires things from section 16.9)

Optional Video: Volume of a Polygon

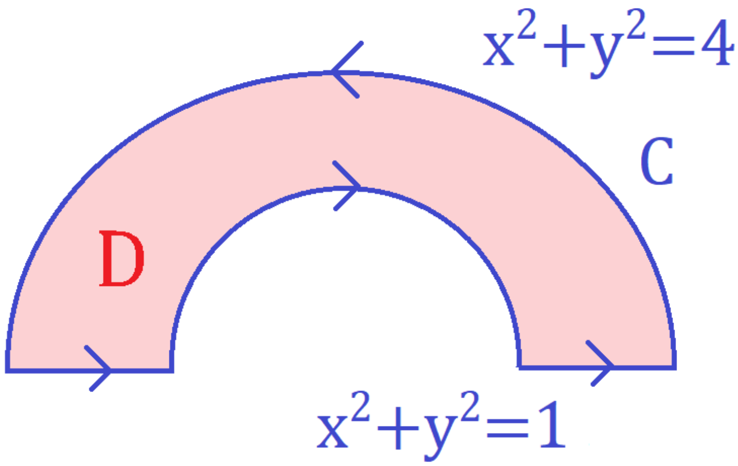
7. More Practice

Example 5: (extra practice)

Calculate the following line integral, where C is the boundary of the region $1 \leq x^2 + y^2 \leq 4$ in the upper-half-plane

$$\int_C y^2 dx + 3xy dy$$

STEP 1: Picture:

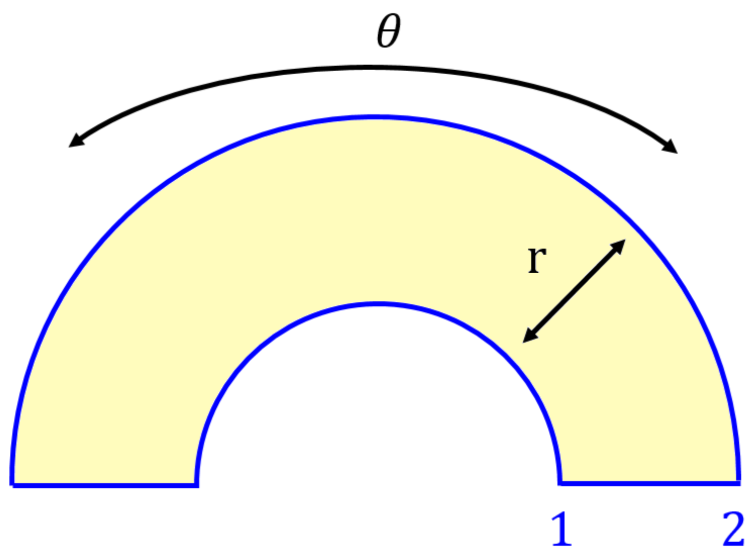


Remark: Notice that the small circle in the middle is in the *clockwise* direction. That is ok, because D is still to your left!

STEP 2: Integrate: Here $F = \langle y^2, 3xy \rangle$

$$\begin{aligned}
&\int_C y^2 dx + 3xy dy \\
&= \int\int_D (3xy)_x - \left(y^2\right)_y dxdy \\
&= \int\int_D 3y - 2y \, dxdy \\
&= \int\int_D y \, dxdy
\end{aligned}$$

STEP 3:



$$\begin{cases} 1 \leq r \leq 2 \\ 0 \leq \theta \leq \pi \end{cases}$$

$$
\begin{aligned}
&= \int_0^{\pi} \int_1^2 r \sin(\theta) r dr d\theta \\
&= \left(\int_1^2 r^2 dr \right) \left(\int_0^{\pi} \sin(\theta) d\theta \right) \\
&= \left(\frac{7}{3} \right) (2) \\
&= \frac{14}{3}
\end{aligned}
$$