# LECTURE 37: CURL AND DIVERGENCE

**Today:** Two new cool operations that you can do with vector fields: Curl and Divergence!

## 1. Divergence

**Divergence**

If $F = \langle P, Q, R \rangle$, then $\text{div}(F) = P_x + Q_y + R_z$

**Example 1:**

If $F = \left\langle x^2, y^2, z^2 \right\rangle$ then

$$\text{div}(F) = \left(x^2\right)_x + \left(y^2\right)_y + \left(z^2\right)_z = \underbrace{2x + 2y + 2z}_{\text{A Number}}$$

**Example 2:**

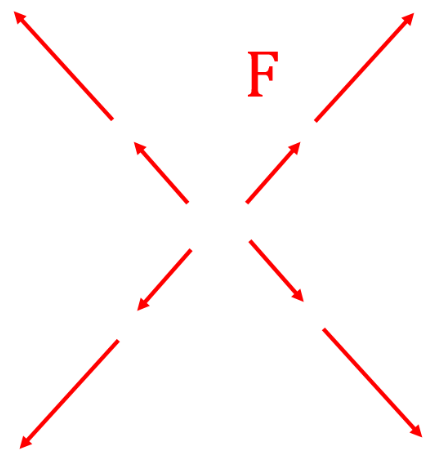
If $F = \left\langle \tan^{-1}(xz), e^{yz}, \ln(1+xz) \right\rangle$, then

$$\begin{aligned} \text{div}(F) &= \left(\tan^{-1}(xz)\right)_x + \left(e^{yz}\right)_y + \left(\ln(1+xz)\right)_z \\ &= \left(\frac{1}{(xz)^2+1}\right) z + e^{yz}\,(z) + \left(\frac{1}{1+xz}\right) x \end{aligned}$$

**Interpretation:** $\text{div}(F)$ measures the **expansion** of $F$

*Date*: Monday, November 22, 2021.

**Example:** $\text{div}(\langle x, y, z\rangle) = 1 + 1 + 1 = 3$




$F$ "expands" at a rate of 3

**In fact:** If $\text{div}(F) = 0$, then $F$ is *incompressible* (= non-expanding)

## 2. Motivation for Curl

**Goal**

What is the analog of $Q_x - P_y$ in 3 dimensions?

Suppose $F = \langle P, Q, R\rangle$ is conservative, that is $F = \nabla f$

$$\langle P, Q, R\rangle = \langle f_x, f_y, f_z\rangle$$

So by Clairaut:

$$\begin{aligned}
f_{xy} =& f_{yx} \Rightarrow (f_x)_y = (f_y)_x \Rightarrow P_y = Q_x \Rightarrow Q_x - P_y = 0 \\
f_{yz} =& f_{zy} \Rightarrow (f_y)_z = (f_z)_y \Rightarrow Q_z = R_y \Rightarrow R_y - Q_z = 0 \\
f_{xz} =& f_{zx} \Rightarrow (f_x)_z = (f_z)_x \Rightarrow P_z = R_x \Rightarrow P_z - R_x = 0
\end{aligned}$$

The amazing thing is that there is **one** operation called the *curl* that takes care of all **three** cases at once.

## 3. Curl

**Definition**

$$\operatorname{curl}(F) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

**Example 3:**

Find $\operatorname{curl}(F)$, where $F = \langle 0, -z, y \rangle$

$$\begin{aligned} \operatorname{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -z & y \end{vmatrix} \\ &= \left\langle \frac{\partial}{\partial y}(y) - \frac{\partial}{\partial z}(-z), -\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial z}(0), \frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial y}(0) \right\rangle \\ &= \langle 1+1, 0, 0 \rangle \\ &= \langle 2, 0, 0 \rangle \end{aligned}$$

**Remark:** $\operatorname{curl}(F)$ is a vector, not a number! (as opposed to $\operatorname{div}(F)$, which is a number)

**Example 4:**

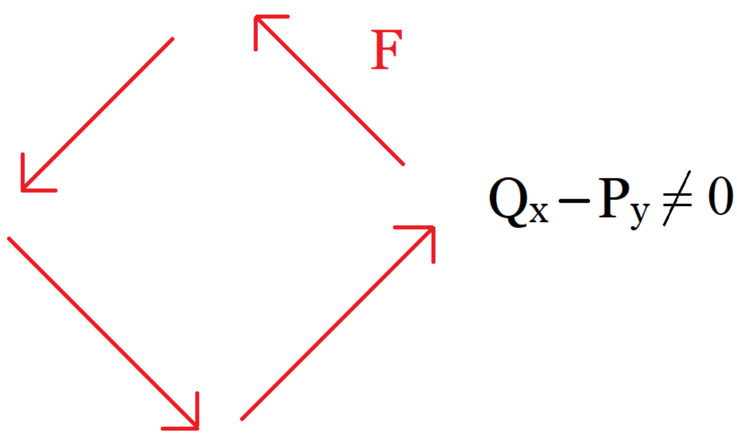
Find $\operatorname{curl} F$, where $F = \langle xz, yz, xy \rangle$

$$\begin{aligned}
\operatorname{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & xy \end{vmatrix} \\
&= \left\langle \frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(yz), -\frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial z}(xz), \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xz) \right\rangle \\
&= \langle x - y, -y + x, 0 \rangle
\end{aligned}$$

## 4. Interpretation

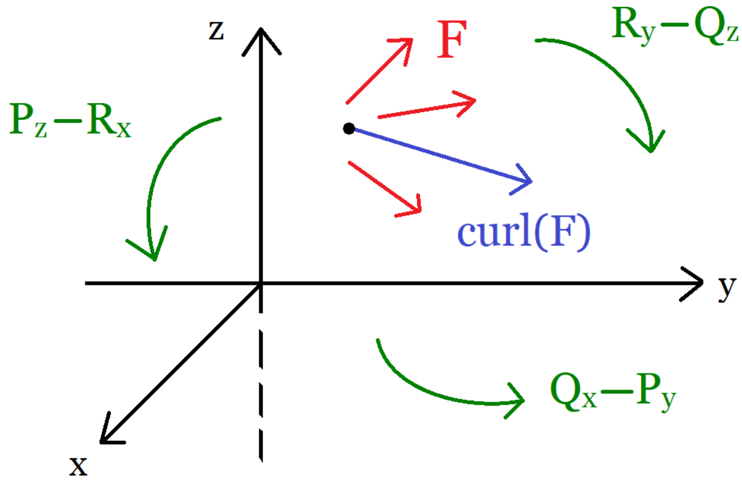
**Intuitively:** $\operatorname{curl}(F)$ measures the **rotation** of $F$.

**Recall:** In $2D$, $Q_x - P_y$ measures the microscopic rotation of $F$




Here we have a 3 dimensional version of this phenomenon:

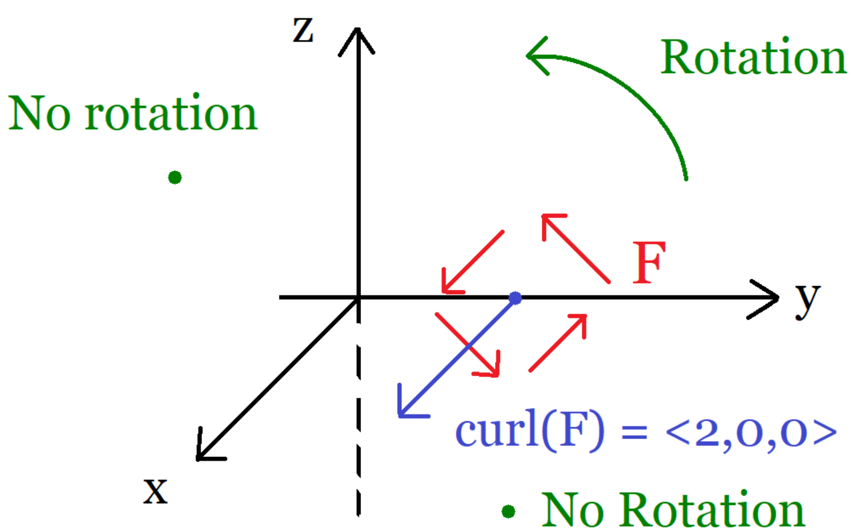
$$\operatorname{curl}(F) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$




So curl$(F)$ measures how $F$ rotates, but in each plane.

**Example 5:**

$F = \langle 0, -z, y \rangle$, showed curl$(F) = \langle 2, 0, 0 \rangle$



Here $\text{curl}(F)$ is the axis of rotation of $F$

## 5. Conservative Vector Fields

The most important thing about curl is that it gives us a very elegant way of checking whether a vector field is conservative or not.

**Fact:**

If $F$ is conservative, then $\text{curl}(F) = \langle 0, 0, 0 \rangle$

**Why?** We showed that if $F = \langle P, Q, R \rangle$ is conservative, then

$$\begin{cases} Q_x - P_y = 0 \\ R_y - Q_z = 0 \\ P_z - R_x = 0 \end{cases}$$

$$\text{Therefore: } \text{curl}(F) = \left\langle \underbrace{R_y - Q_z}_{0}, \underbrace{P_z - R_x}_{0}, \underbrace{Q_x - P_y}_{0} \right\rangle = \langle 0, 0, 0 \rangle$$

**Conversely:** If $\text{curl}(F) = \langle 0, 0, 0 \rangle$ (and no holes), then $F$ is conservative (this uses Stokes' theorem, see section 16.8)

**Important Fact:**

$$F \text{ conservative } \Leftrightarrow \text{curl}(F) = \langle 0, 0, 0 \rangle$$

So this is a good test for conservative in 3 dimensions, 3D analog of $P_y = Q_x$

**Interpretation:** Conservative vector fields are **irrotational** (curl is zero), just like in 2 dimensions.

**Example 6: (Good exam question)**

(a) Is $F = \left\langle y^2z^3, 2xyz^3, 3xy^2z^2 \right\rangle$ conservative?

$$\begin{aligned}
\operatorname{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} \\
&= \left\langle \frac{\partial}{\partial y}(3xy^2z^2) - \frac{\partial}{\partial z}(2xyz^3), -\frac{\partial}{\partial x}(3xy^2z^2) + \frac{\partial}{\partial z}(y^2z^3), \right. \\
&\quad \left. \frac{\partial}{\partial x}(2xyz^3) - \frac{\partial}{\partial y}(y^2z^3) \right\rangle \\
&= \left\langle \cancel{6xyz^2} - \cancel{6xyz^2}, \cancel{-3y^2z^2} + \cancel{3y^2z^2}, \cancel{2yz^3} - \cancel{2yz^3} \right\rangle \\
&= \langle 0, 0, 0 \rangle \qquad \textit{BINGO}!
\end{aligned}$$

**Answer:** Yes

(b) Find $f$ such that $F = \nabla f$

$$\left\langle y^2z^3, 2xyz^3, 3xy^2z^2 \right\rangle = \langle f_x, f_y, f_z \rangle$$

$$\begin{aligned}
f_x = y^2z^3 &\Rightarrow f = \int y^2z^3 dx = xy^2z^3 + \text{ JUNK} \\
f_y = 2xyz^3 &\Rightarrow f = \int 2xyz^3 dy = xy^2z^3 + \text{ JUNK} \\
f_z = 3xy^2z^2 &\Rightarrow f = \int 3xy^2z^2 dz = xy^2z^3 + \text{ JUNK}
\end{aligned}$$

$$f(x, y, z) = xy^2z^3$$

(c) Evaluate $\int_C F \cdot dr$, $C$ any path connecting $(0, 0, 0)$ and $(1, 1, 1)$

$$\begin{aligned}\int_C F \cdot dr =& f(1,1,1) - f(0,0,0)\\ =&(1)(1^2)(1^3) - (0)(0^2)(0^3)\\ =&1\end{aligned}$$

## 6. Div, Grad, Curl

**Question:** How are $\text{div}(F)$, $\text{curl}(F)$, and $\nabla f$ related?

**Important Facts:**

(1) $\text{curl}(\nabla f) = \langle 0, 0, 0\rangle$

(2) $\text{div}(\text{curl}(F)) = 0$


**Why?**

(1) Direct calculation, or: $\nabla f$ is conservative, so $\text{curl}(\nabla f) = \langle 0, 0, 0\rangle$ (by fact above)

(2) Direct calculation

**Mnemonic:** If you follow the book's order, then

$$\text{New Topic (Topic before that)} = 0$$

$$\underbrace{\text{curl}}_{16.5}\underbrace{\nabla f}_{14.6} = \langle 0,0,0\rangle$$

$$\underbrace{\text{div}}_{16.5 \text{ Part } 2}\quad \underbrace{\text{curl}\, F}_{16.5 \text{ Part } 1} = 0$$

**Example 7:**

Can $F = \left\langle xz, xyz, -y^2 \right\rangle$ be written as $\text{curl}\, G$ for some $G$?

**No!** Suppose $F = \text{curl}\, G$, then

$$\text{div}(F) = \text{div}(\text{curl}\, G) = 0 \qquad \text{(By Fact)}$$

$$\text{But:}\quad \text{div}(F) = (xz)_x + (xyz)_y + (-y^2)_z = z + xz \neq 0$$

So $0 \neq 0$, which is a (juicy) contradiction

**Warning:**

$$\text{div}(\nabla f) \neq 0$$

$$\begin{aligned}\text{div}(\nabla f) &= \text{div}\left(\langle f_x, f_y, f_z\rangle\right)\\ &= (f_x)_x + (f_y)_y + (f_z)_z\\ &= f_{xx} + f_{yy} + f_{zz}\\ &= \Delta f \text{ (Laplacian of } f\text{, from 14.3)}\end{aligned}$$