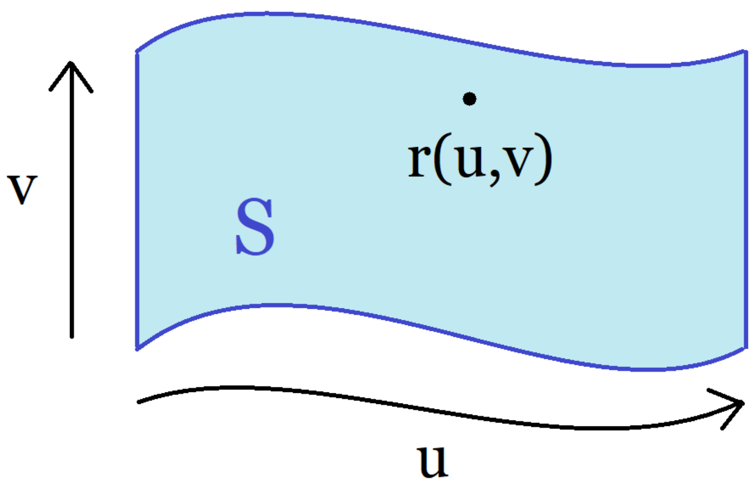
LECTURE 38: PARAMETRIC SURFACES (I)

Video: Parametric Surfaces

The goal for the rest of the course is to generalize everything that we know about line integrals to surfaces.

Goal: How to parametrize a surface S ?



Since S is 2-dimensional, we need 2 parameters u and v. The analog of $r(t)$ is then $r(u, v)$.

Note: Think of this like an Etch-a-sketch, where one direction is u and the other one is v:

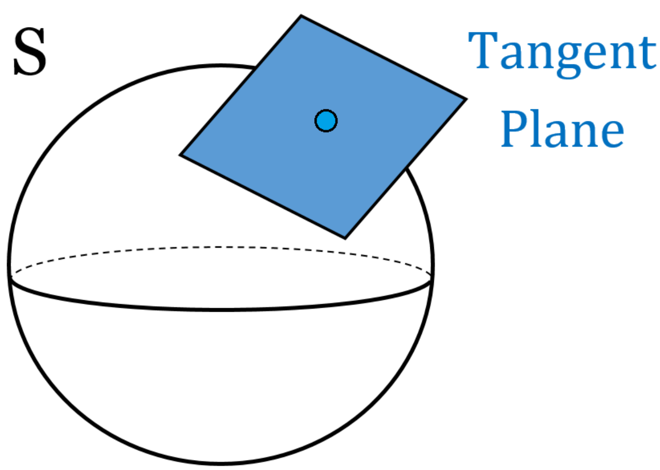
Date: Monday, November 29, 2021.

Note: In Appendix 1 of those notes, you can find examples of parametric surfaces. Please make sure to review at least the first three examples.

1. Tangent Planes

Video: Tangent Plane to a Surface

Goal: Find the tangent plane to a (parametric) surface, similar to what we did when we covered partial derivatives.



Example 1:

Find the equation of the tangent plane to the following surface at $u = 1, v = 2$:

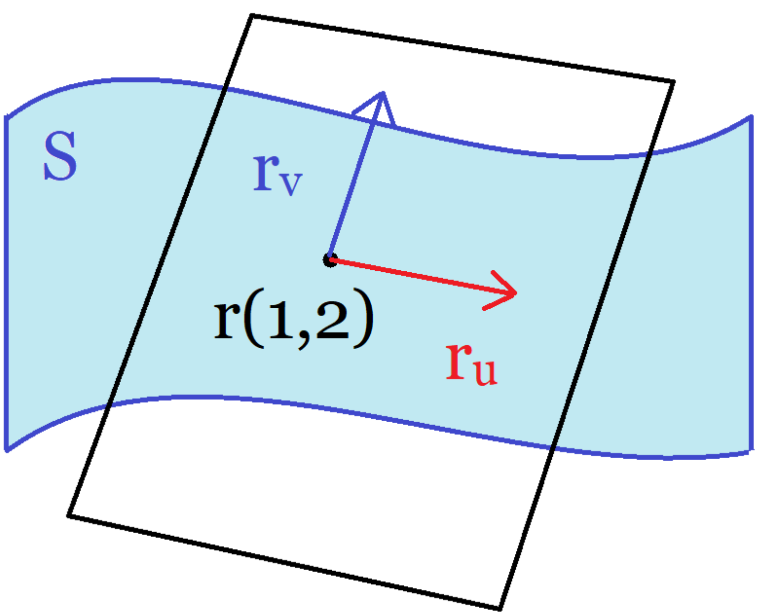
$$r(u, v) = \left\langle u^2 + 1, v^2 + 1, u + v \right\rangle$$

STEP 1: Partial Derivatives

Calculate r_u and r_v (at $u = 1, v = 2$)

$$r_u = \left\langle (u^2 + 1)_u, (v^2 + 1)_u, (u + v)_u \right\rangle = \langle 2u, 0, 1\rangle = \langle 2, 0, 1\rangle \text{ (at } u = 1, v = 2)$$
$$r_v = \left\langle (u^2 + 1)_v, (v^2 + 1)_v, (u + v)_v \right\rangle = \langle 0, 2v, 1\rangle = \langle 0, 4, 1\rangle$$

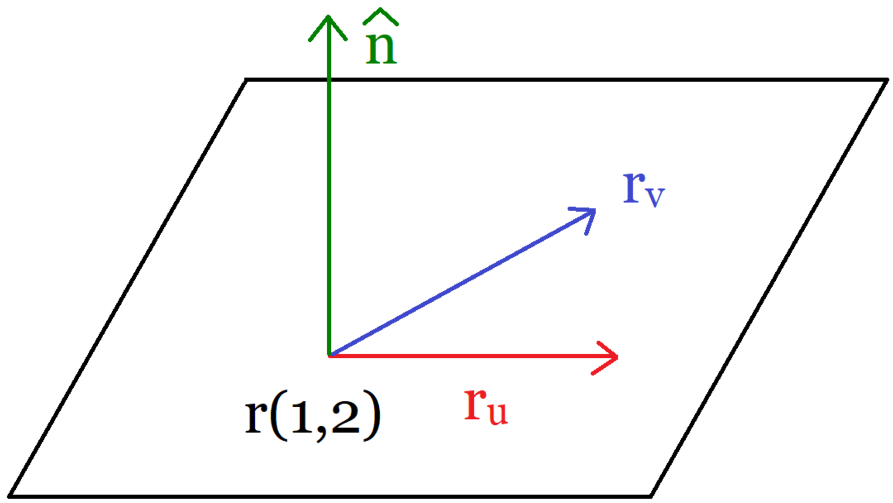
Fact: r_u and r_v are on the tangent plane



Recall:

To find equation of a plane, need a **point** and a **normal vector**

STEP 2: Normal Vector:



$$\begin{aligned}
\hat{n} =& r_u \times r_v \\
=& \langle 2, 0, 1\rangle \times \langle 0, 4, 1\rangle \\
=& \begin{vmatrix} i & j & k \\ 2 & 0 & 1 \\ 0 & 4 & 1 \end{vmatrix} \\
=& \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} k \\
=& -4i - 2j + 8k \\
=& \langle -4, -2, 8\rangle
\end{aligned}$$

STEP 3: Point:

$$r(1,2) = \left\langle 1^2 + 1, 2^2 + 1, 1 + 2 \right\rangle = \langle 2, 5, 3\rangle$$

STEP 4: Equation: $\hat{n} = \langle -4, -2, 8 \rangle$, Point $(2, 5, 3)$

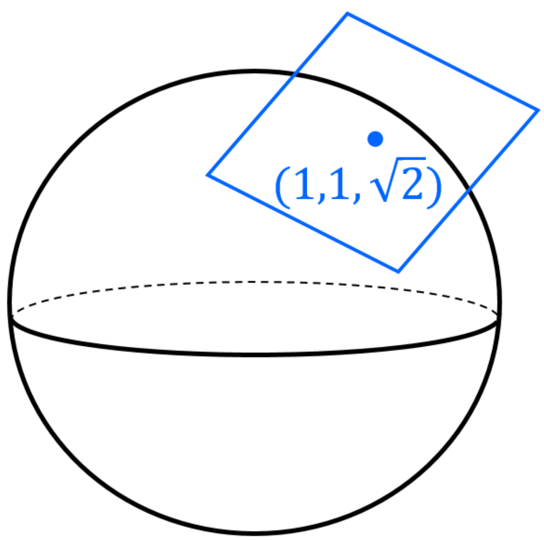
$$-4(x-2) - 2(y-5) + 8(z-3) = 0$$

⚠ Don't confuse $\hat{n}$ (normal vector) with $n = \frac{\hat{n}}{\|\hat{n}\|}$ (**unit** normal vector)

Example 2: (extra practice)

Find the tangent plane to the sphere $x^2 + y^2 + z^2 = 4$ at $(1, 1, \sqrt{2})$

STEP 1: Picture:



STEP 2: Parametrize S:

$$\begin{cases} x = 2\sin(\phi)\cos(\theta) \\ y = 2\sin(\phi)\sin(\theta) \\ z = 2\cos(\phi) \end{cases} \implies \begin{cases} r(\theta, \phi) = \langle 2\sin(\phi)\cos(\theta), 2\sin(\phi)\sin(\theta), 2\cos(\phi) \rangle \\ \quad 0 \le \theta \le 2\pi \\ \quad 0 \le \phi \le \pi \end{cases}$$

STEP 3: Find θ and ϕ (analog of $u = 1$ and $v = 2$)

$$(2\sin(\phi)\cos(\theta), 2\sin(\phi)\sin(\theta), 2\cos(\phi)) = (1, 1, \sqrt{2})$$

The last equation gives you:

$$2\cos(\phi) = \sqrt{2} \Rightarrow \cos(\phi) = \frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}$$

Then the first equation becomes

$$\begin{aligned} 2\sin(\phi)\cos(\theta) &= 1 \\ 2\sin\left(\frac{\pi}{4}\right)\cos(\theta) &= 1 \\ 2\left(\frac{\sqrt{2}}{2}\right)\cos(\theta) &= 1 \\ \sqrt{2}\cos(\theta) &= 1 \\ \cos(\theta) &= \frac{1}{\sqrt{2}} \\ \theta &= \frac{\pi}{4} \end{aligned}$$

Therefore $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{4}$

STEP 4: Partial Derivatives

$$\begin{aligned} r_\theta &= \langle -2\sin(\phi)\sin(\theta), 2\sin(\phi)\cos(\theta), 0\rangle \text{ (Use } \theta = \frac{\pi}{4}, \phi = \frac{\pi}{4}) \\ &= \left\langle -2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right), 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right), 0\right\rangle \\ &= \langle -1, 1, 0\rangle \end{aligned}$$

$$\begin{aligned} r_\phi &= \langle 2\cos(\phi)\cos(\theta), 2\cos(\phi)\sin(\theta), -2\sin(\phi)\rangle \\ &= \left\langle 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right), 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right), -2\left(\frac{1}{\sqrt{2}}\right)\right\rangle \\ &= \left\langle 1, 1, -\sqrt{2}\right\rangle \end{aligned}$$

STEP 5: Normal Vector

$$\hat{n} = r_\theta \times r_\phi = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 1 & 1 & -\sqrt{2} \end{vmatrix} = \left\langle -\sqrt{2}, -\sqrt{2}, -2 \right\rangle$$

STEP 6: Point: $(1, 1, \sqrt{2})$

STEP 7: Equation:

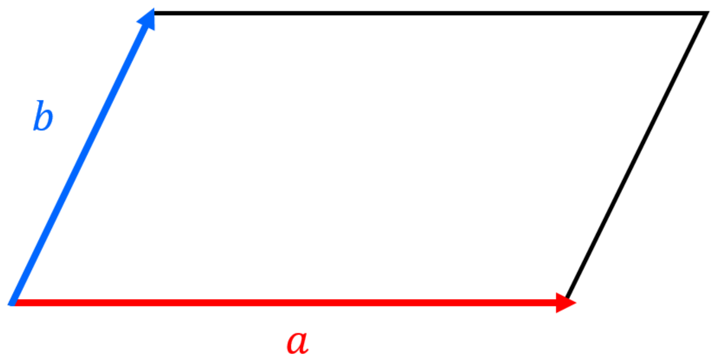
$$-\sqrt{2}(x-1) - \sqrt{2}(y-1) - 2(z-\sqrt{2}) = 0$$

2. Quick Facts

For our second application of parametric surfaces, we'll calculate the area of **any** surface! (yes, *any* surface)

Fact 1: Area of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ is:

$$\|\mathbf{a} \times \mathbf{b}\|$$

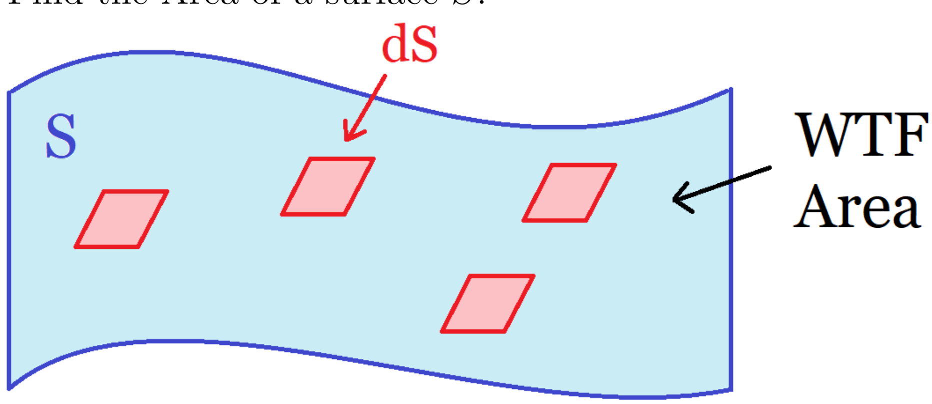


Fact 2: If $c, d > 0$, then

$$\|(c\mathbf{a}) \times (d\mathbf{b})\| = \|\mathbf{a} \times \mathbf{b}\| \, cd$$

3. Surface Areas

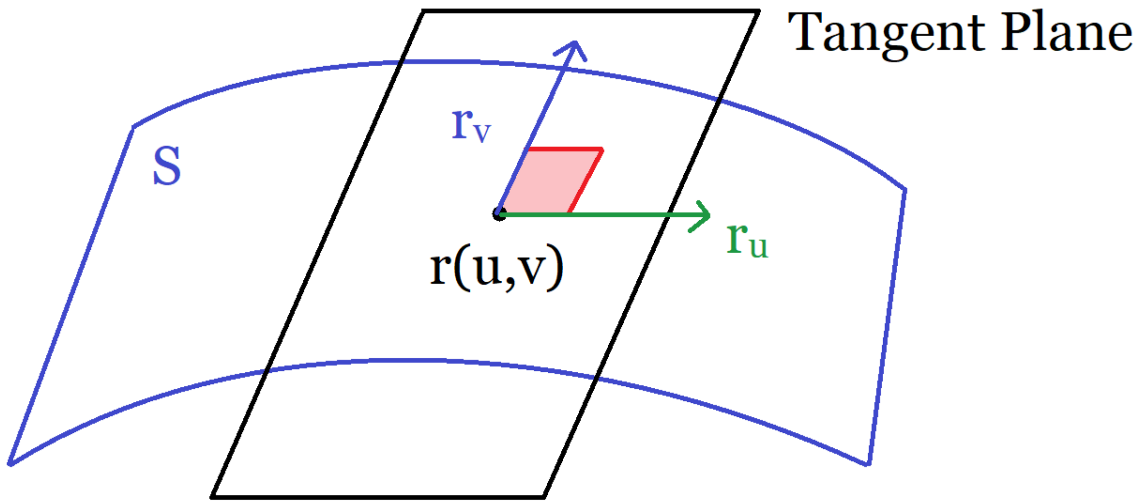
Goal: Find the Area of a surface S:



STEP 1: Cover S with mini parallelograms

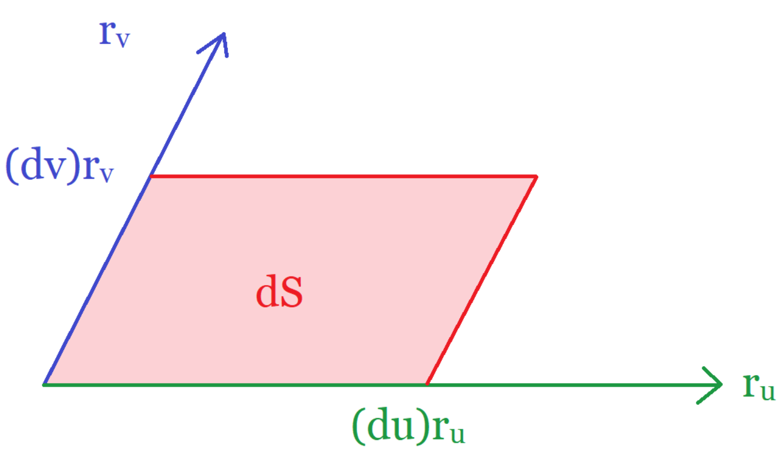
Recall:

r_u and r_v (partial derivatives) are on the tangent plane



First Try: The parallelogram with sides r_u and r_v, but this is too big!

Solution: Rescale it to have sides $r_u \underbrace{(du)}_{\text{Small}}$ and $r_v \underbrace{(dv)}_{\text{Small}}$:



STEP 2: Find the area of each mini-parallelogram

$$\begin{aligned} dS &= \text{Area(Parallelogram)} \\ &= \|r_u(du) \times r_v(dv)\| \ \text{(By Fact 1)} \\ &= \|r_u \times r_v\|\, dudv \ \text{(By Fact 2)} \end{aligned}$$

STEP 3: Integrate (sum up)

Surface Area (memorize)

$$\text{Area }(S) = \int\int dS = \int\int_D \|r_u \times r_v\|\, dudv$$

(Think of it as sum of mini areas)

4. Example

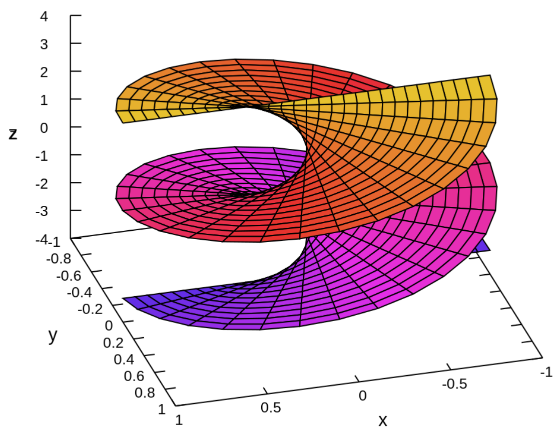
Example 3:

Find Area(S), where S : Helicoid with equations

$$\begin{aligned} r(u,v) &= \langle u\cos(v), u\sin(v), v\rangle \\ 0 &\leq u \leq 1 \\ 0 &\leq v \leq \pi \end{aligned}$$

STEP 1: Picture:[1]

[1]Taken from Wikipedia



STEP 2: Derivatives:

$$r_u = \langle \cos(v), \sin(v), 0 \rangle$$
$$r_v = \langle -u\sin(v), u\cos(v), 1 \rangle$$

STEP 3: Cross Product:

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 1 \end{vmatrix}$$
$$= \left\langle \sin(v), -\cos(v), u\cos^2(v) + u\sin^2(v) \right\rangle$$
$$= \langle \sin(v), -\cos(v), u \rangle$$

STEP 4: dS :

$$dS = \|r_u \times r_v\| = \sqrt{\sin^2(v) + \cos^2(v) + u^2} = \sqrt{1 + u^2}$$

STEP 5: Integrate:

$$\begin{aligned}\text{Area } (S) &= \int\int dS \\ &= \int_0^{\pi}\int_0^1 \sqrt{u^2+1}\,du dv \\ &= \pi\int_0^1 \sqrt{u^2+1}\,du \text{ (Use } u=\tan(\theta)) \\ &= \text{(See Appendix 2)} \\ &= \frac{\pi}{2}\left(\sqrt{2}+\ln(\sqrt{2}+1)\right)\end{aligned}$$

Note: Problems like those require you to know how to do trig substitution, check out the following videos for review:

Video 1: Integral of $\sqrt{x^2+1}$
Video 2: Integral of $\sqrt{1-x^2}$
Video 3: Integral of $\sqrt{x^2-1}$

Note: I probably won't ask about trig substitution on the quizzes or exams (at least not crazy ones like the one below) but you might have to do that on the homework.

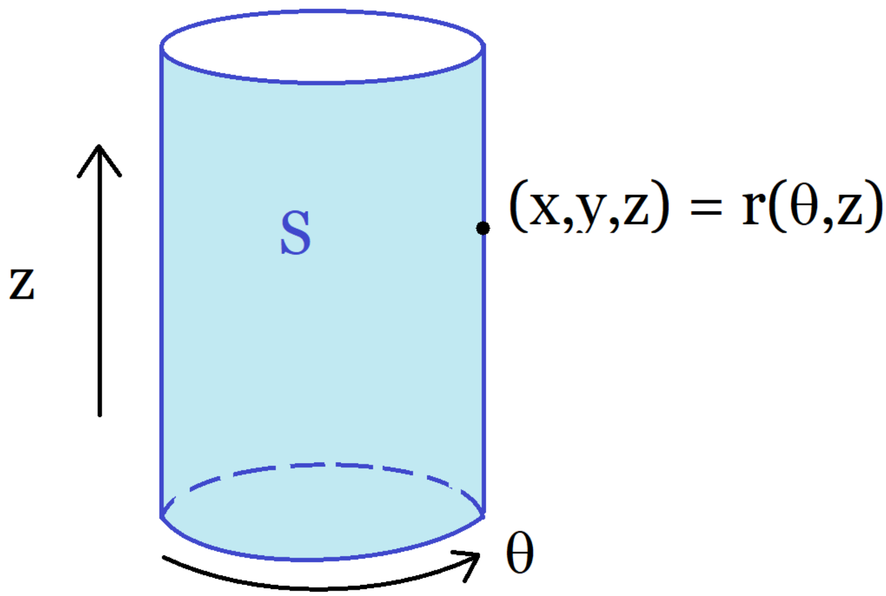
5. Appendix 1: Parametric Surfaces

Video: Parametric Surfaces

Example 4:

Parametrize the cylinder $x^2+y^2=4$

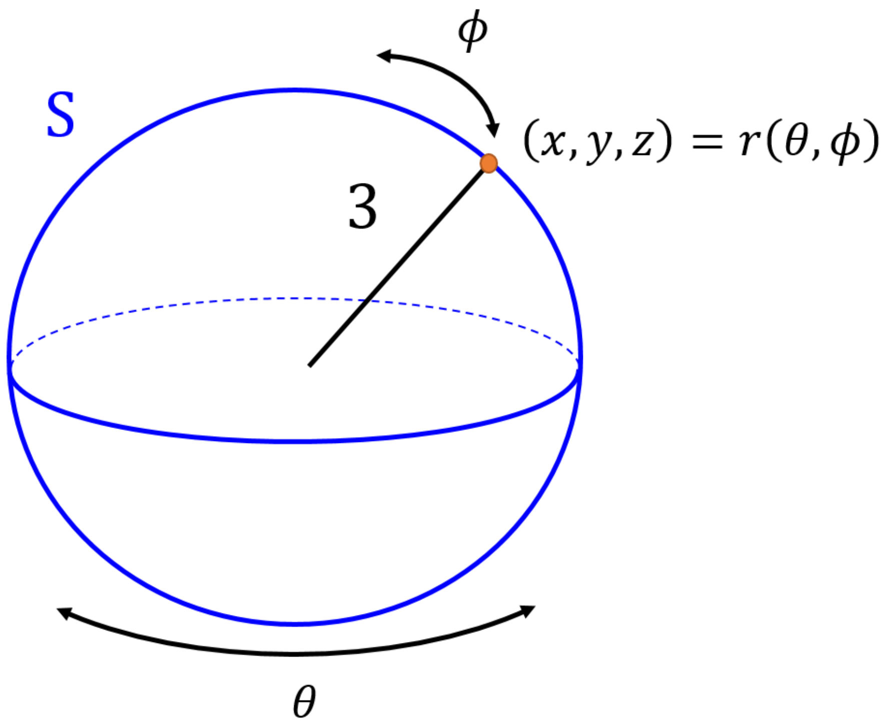
It's just cylindrical coordinates with $r=2$



$$\begin{cases} x = 2\cos(\theta) \\ y = 2\sin(\theta) \\ z = z \end{cases} \implies \begin{cases} r(\theta, z) = \langle 2\cos(\theta), 2\sin(\theta), z\rangle \\ 0 \leq \theta \leq 2\pi \\ -\infty < z < \infty \end{cases}$$

Example 5:

Parametrize the sphere $x^2 + y^2 + z^2 = 9$

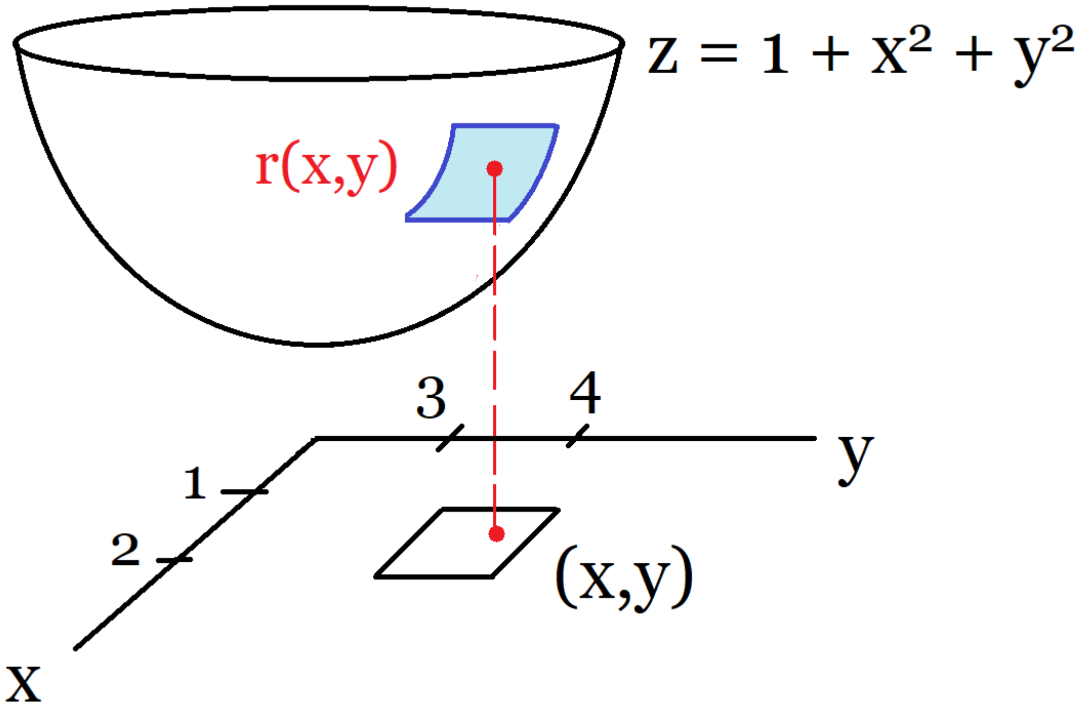


$$\begin{cases} x = & 3\sin(\phi)\cos(\theta) \\ y = & 3\sin(\phi)\sin(\theta) \\ z = & 3\cos(\phi) \end{cases}$$

$$\begin{cases} r(\theta, \phi) = \langle 3\sin(\phi)\cos(\theta), 3\sin(\phi)\sin(\theta), 3\cos(\phi)\rangle \\ \quad 0 \leq \theta \leq 2\pi \\ \quad 0 \leq \phi \leq \pi \end{cases}$$

Example 6: Functions

Parametrize the portion of the paraboloid $z = 1 + x^2 + y^2$ over the rectangle $[1, 2] \times [3, 4]$



$$\begin{cases} x = x \\ y = y \\ z = 1 + x^2 + y^2 \end{cases} \Longrightarrow \begin{cases} r(x,y) = \left\langle x, y, 1 + x^2 + y^2 \right\rangle \\ 1 \leq x \leq 2 \\ 3 \leq y \leq 4 \end{cases}$$

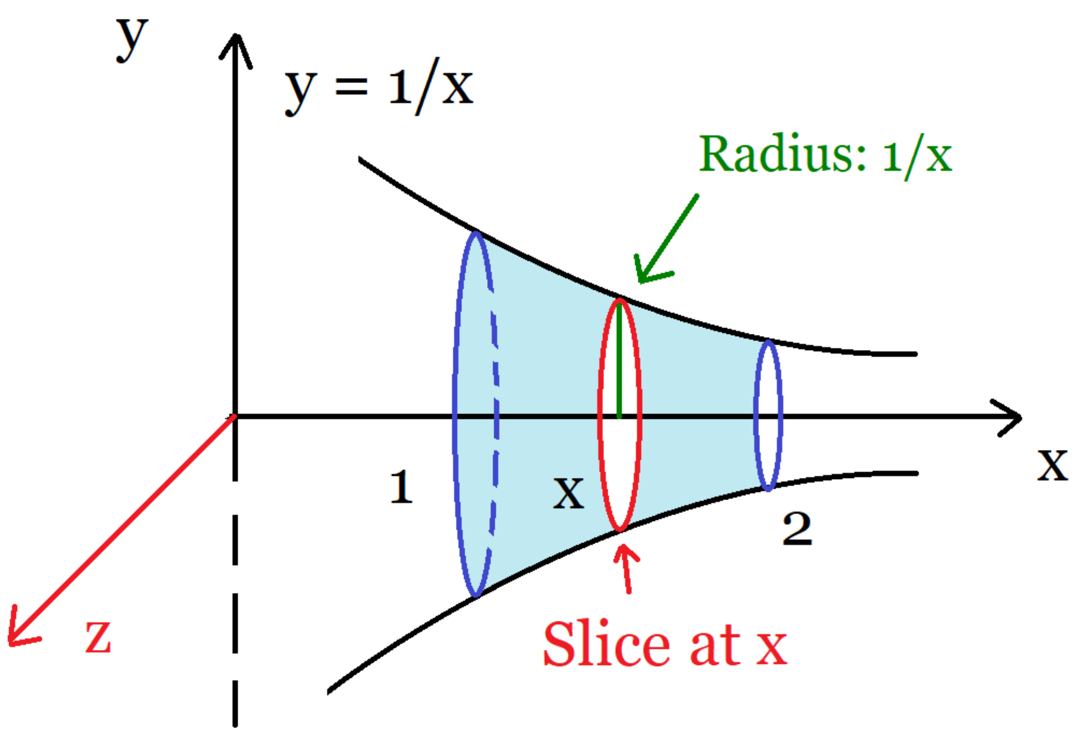
More generally, for functions $z = f(x, y)$

$$r(x, y) = \langle x, y, f(x, y)\rangle$$

(So surfaces are more general than functions)

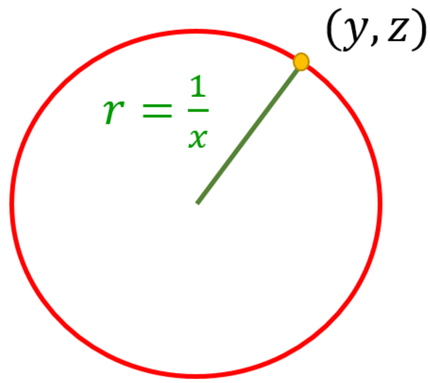
Example 7: Solids of Revolution

(Calculus) Parametrize the Surface obtained by rotating the curve $y = \frac{1}{x}$ between $x = 1$ and $x = 2$ about the $x-$axis



Start with $x = x$, $1 \leq x \leq 2$.

Look at slice at x:



$$\begin{cases} y = & r\cos(\theta) = \left(\frac{1}{x}\right)\cos(\theta) \\ z = & r\sin(\theta) = \left(\frac{1}{x}\right)\sin(\theta) \end{cases}$$

$$\begin{cases} r(x,\theta) = \left\langle x, \left(\frac{1}{x}\right)\cos(\theta), \left(\frac{1}{x}\right)\sin(\theta)\right\rangle \\ \qquad 1 \leq x \leq 2 \\ \qquad 0 \leq \theta \leq 2\pi \end{cases}$$

More generally: If you rotate the function $f(x)$ about the $x-$axis, then

$$r(x,\theta) = \langle x, f(x)\cos(\theta), f(x)\sin(\theta)\rangle$$

6. Appendix 2: Trig Substitution

Here I give a quick review of how to use trig substitution from Calc 2:

Example 8:

Evaluate the following integral:

$$\int_0^1 \sqrt{u^2+1}\, du$$

STEP 1: Let $u = \tan(\theta)$, then:

(1) $du = \sec^2(\theta)d\theta$

(2) $\sqrt{u^2+1} = \sqrt{\tan^2(\theta)+1} = \sqrt{\sec^2(\theta)} = \sec(\theta)$

(3) $u = 0 \Rightarrow 0 = \tan(\theta) \Rightarrow \theta = 0$

(4) $u = 1 \Rightarrow 1 = \tan(\theta) \Rightarrow \theta = \frac{\pi}{4}$

Therefore our integral becomes:

$$\int_0^1 \sqrt{u^2+1}du = \int_0^{\frac{\pi}{4}} \sec(\theta)\sec^2(\theta)d\theta = \int_0^{\frac{\pi}{4}} \sec^3(\theta)d\theta$$

STEP 2: Integrate by parts:

$$\begin{aligned} u &= \sec(\theta) & dv &= \sec^2(\theta)d\theta \\ du &= \sec(\theta)\tan(\theta)d\theta & v &= \tan(\theta) \end{aligned}$$

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \sec^3(\theta)d\theta &= \int_0^{\frac{\pi}{4}} \sec(\theta)\sec^2(\theta)d\theta \\
&= [\sec(\theta)\tan(\theta)]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec(\theta)\tan(\theta)\tan(\theta)d\theta \\
&= \sqrt{2}(1) - (1)(0) - \int_0^{\frac{\pi}{4}} \sec(\theta)\tan^2(\theta)d\theta \\
&= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec(\theta)\left(\sec^2(\theta) - 1\right)d\theta \\
&= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3(\theta)d\theta + \int_0^{\frac{\pi}{4}} \sec(\theta)d\theta \\
&= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3(\theta)d\theta + [\ln|\sec(\theta)+\tan(\theta)|]_0^{\frac{\pi}{4}} \\
&= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3(\theta)d\theta + \ln\left(\sqrt{2}+1\right) - \ln(1+0) \\
&= \sqrt{2} + \ln\left(\sqrt{2}+1\right) - \int_0^{\frac{\pi}{4}} \sec^3(\theta)d\theta
\end{aligned}
$$

STEP 3: Solving for our integral, we finally get:

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \sec^3(\theta)d\theta &= \sqrt{2} + \ln\left(\sqrt{2}+1\right) - \int_0^{\frac{\pi}{4}} \sec^3(\theta)d\theta \\
2\int_0^{\frac{\pi}{4}} \sec^3(\theta)d\theta &= \sqrt{2} + \ln\left(\sqrt{2}+1\right) \\
\int_0^{\frac{\pi}{4}} \sec^3(\theta)d\theta &= \frac{1}{2}\left(\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right) \\
\int_0^1 \sqrt{u^2+1}du &= \frac{1}{2}\left(\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right)
\end{aligned}
$$