LECTURE 39: PARAMETRIC (II) + SURFACE INTEGRALS (I)

1. Surface Area of a Sphere

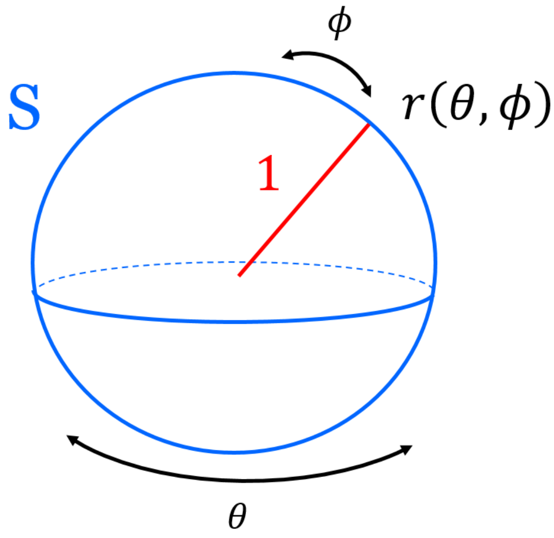
Video: Surface Area of a Sphere

OMG Example 1:

Find the surface area of a sphere of radius 1

All those years you've been told that it's $4\pi r^2$, but now I can finally show you why it's true!

STEP 1: Picture:



Date: Wednesday, December 1, 2021.

STEP 2: Parametrize: Basically spherical coordinates with $\rho = 1$

$$\begin{cases} r(\theta, \phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle \\ \quad 0 \leq \theta \leq 2\pi \\ \quad 0 \leq \phi \leq \pi \end{cases}$$

STEP 3: Derivatives:

$$r_\theta = \langle -\sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta), 0 \rangle$$
$$r_\phi = \langle \cos(\phi)\cos(\theta), \cos(\phi)\sin(\theta), -\sin(\phi) \rangle$$

STEP 4: Cross Product:

$$\begin{aligned} r_\theta \times r_\phi &= \begin{vmatrix} i & j & k \\ -\sin(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) & 0 \\ \cos(\phi)\cos(\theta) & \cos(\phi)\sin(\theta) & -\sin(\phi) \end{vmatrix} \\ &= \Big\langle -\sin^2(\phi)\cos(\theta), -\sin^2(\phi)\sin(\theta), \\ &\quad -\sin(\phi)\cos(\phi)\sin^2(\theta) - \sin(\phi)\cos(\phi)\cos^2(\theta) \Big\rangle \\ &= \Big\langle -\sin^2(\phi)\cos(\theta), -\sin^2(\phi)\sin(\theta), -\sin(\phi)\cos(\phi) \Big\rangle \end{aligned}$$

STEP 5: dS

$$\begin{aligned} dS &= \|r_\theta \times r_\phi\| \\ &= \left(\sin^4(\phi)\cos^2(\theta) + \sin^4(\phi)\sin^2(\theta) + \sin^2(\phi)\cos^2(\phi)\right)^{\frac{1}{2}} \\ &= \left(\sin^4(\phi) + \sin^2(\phi)\cos^2(\phi)\right)^{\frac{1}{2}} \\ &= \left(\sin^2(\phi)\left(\sin^2(\theta) + \cos^2(\theta)\right)\right)^{\frac{1}{2}} \\ &= \left(\sqrt{\sin^2(\phi)}\right) \\ &= \sin(\phi) \end{aligned}$$

STEP 6: Integrate

$$
\begin{aligned}
\text{Area } (S) &= \int\int dS \\
&= \int_0^{\pi}\int_0^{2\pi} \sin(\phi) d\theta d\phi \\
&= 2\pi \left(\int_0^{\pi} \sin(\phi) d\phi\right) \\
&= (2\pi)(2) \\
&= 4\pi \\
&= 4\pi(1)^2
\end{aligned}
$$

Note: In general, using the same method, you get that the surface area of a sphere of radius r is $4\pi r^2$! WOW

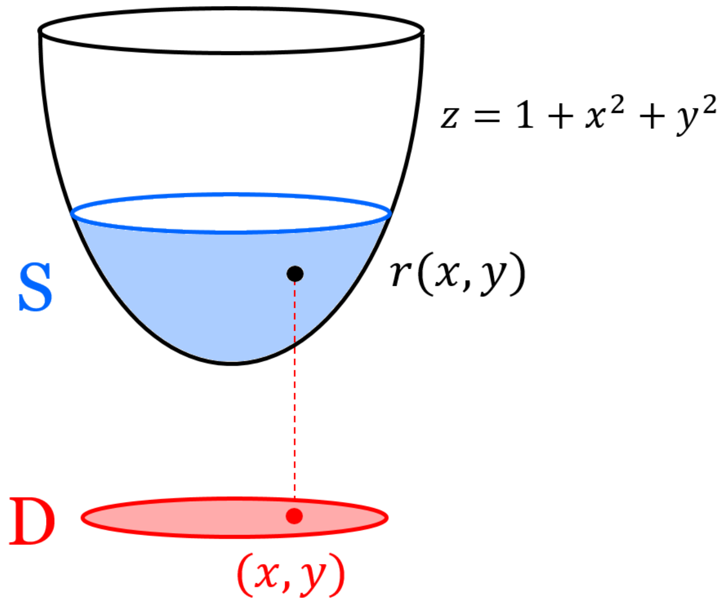
2. The Case of Functions

What if S is the graph of a function? In that case, the surface area formula becomes much simpler!

Example 2:

Find Area(S), where S is the part of the paraboloid $z = 1+x^2+y^2$ over the disk $x^2 + y^2 \leq 4$

STEP 1: Picture:



STEP 2: Parametrize:

$$r(x, y) = \left\langle x, y, 1 + x^2 + y^2 \right\rangle$$

Note: Completely fine to use $r(r, \theta) = \left\langle r\cos(\theta), r\sin(\theta), 1 + r^2 \right\rangle$

STEP 3: Derivatives:

$$r_x = \langle 1, 0, 2x \rangle$$
$$r_y = \langle 0, 1, 2y \rangle$$

STEP 4: Cross Product:

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = \langle -2x, -2y, 1 \rangle = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle$$

STEP 5: dS

$$dS = \|r_x \times r_y\| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

STEP 6: Integrate

$$\begin{aligned}
\text{Area } (S) &= \int\int dS \\
&= \int\int_D \sqrt{4x^2 + 4y^2 + 1} dx dy \\
&= \int_0^{2\pi} \int_0^2 \left(4r^2 + 1\right)^{\frac{1}{2}} r dr d\theta \\
&= 2\pi \left[\left(\frac{1}{8}\right)\left(\frac{2}{3}\right)\left(4r^2+1\right)^{\frac{3}{2}}\right]_0^2 \quad (\text{Or use } u = 4r^2 + 1) \\
&= \frac{\pi}{6}\left((17)^{\frac{3}{2}} - 1\right)
\end{aligned}$$

More Generally

If $z = z(x, y)$ is the graph of a function, then

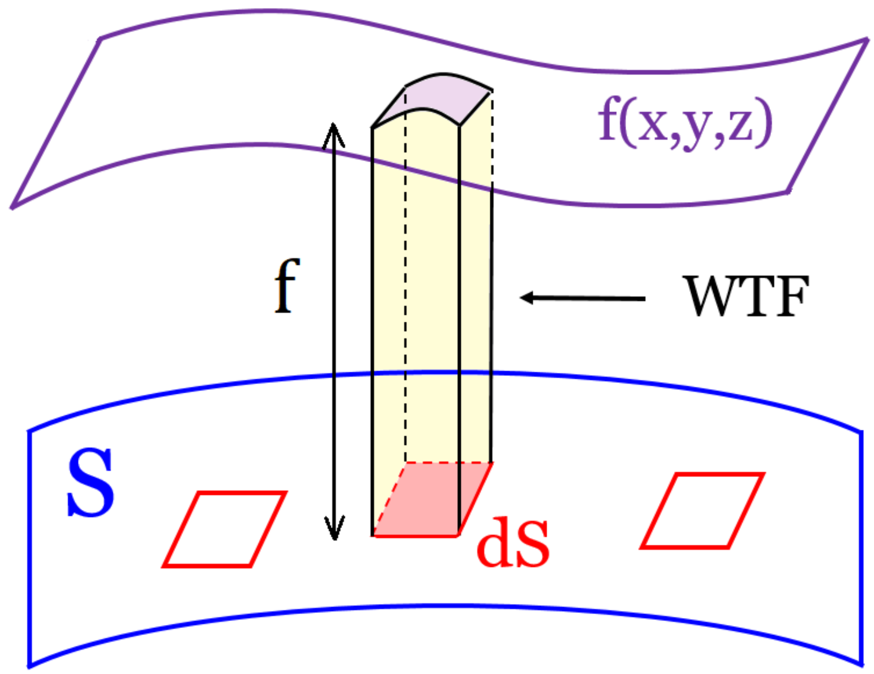
$$\text{Area } (S) = \int\int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy$$

Note: I wouldn't bother memorizing this formula because I'd probably force you to do this problem using parametrizations anyway.

3. Surface Integrals

Video: Surface Integral

Goal: Find the volume of the solid under a function f and over a surface S (think surface of the earth)



Here is where our mini-parallelograms dS help us once again!

Surface Integral

$$\int\int_S f dS = \int\int_D f(r(u,v)) \underbrace{\|r_u \times r_v\| \, dudv}_{dS}$$

This is the same formula as above, except we have $f dS$ instead of dS.

Here Base $= dS$ and Height $= f$, so Volume $=$ Base $\times$ Height $= f dS$

4. Example

Video: Surface Integral

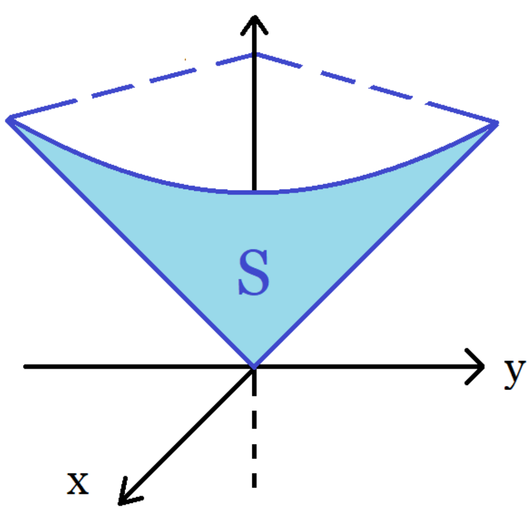
Example 3:

$$\int\int_S y dS$$

S : Cone in the first octant, parametrized by:

$$\begin{aligned} r(u,v) &= \langle u\cos(v), u\sin(v), u\rangle \\ 0 \leq & u \leq 1 \\ 0 \leq & v \leq \frac{\pi}{2} \end{aligned}$$

STEP 1: Picture



STEP 2: Slopes

$$\begin{aligned} r_u &= \langle \cos(v), \sin(v), 1\rangle \\ r_v &= \langle -u\sin(v), u\cos(v), 0\rangle \end{aligned}$$

STEP 3: Normal Vector

$$
\begin{aligned}
r_u \times r_v &= \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 1 \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix} \\
&= \left\langle -u\cos(v), -u\sin(v), u\cos^2(v) + u\sin^2(v) \right\rangle \\
&= \langle -u\cos(v), -u\sin(v), u \rangle
\end{aligned}
$$

STEP 4: dS

$$
dS = \|r_u \times r_v\| = \left(u^2\cos^2(v) + u^2\sin^2(v) + u^2\right)^{\frac{1}{2}} = \sqrt{u^2 + u^2} = \left(\sqrt{2}\right) u
$$

STEP 5: Integrate

$$
\begin{aligned}
&\int\int_S y \, dS \\
=&\int_0^{\frac{\pi}{2}} \int_0^1 u\sin(v) \left(\sqrt{2}\right) u \, du dv \\
=&\sqrt{2} \left(\int_0^1 u^2 du \right) \left(\int_0^{\frac{\pi}{2}} \sin(v) dv \right) \\
=&\left[\frac{\sqrt{2}}{3} u^3 \right]_0^1 \left[-\cos(v) \right]_0^{\frac{\pi}{2}} \\
=&\frac{\sqrt{2}}{3}
\end{aligned}
$$

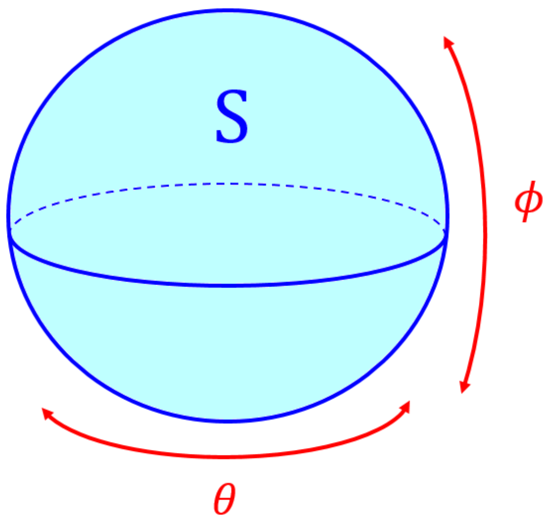
5. Mass

Application: If f is density, then $\int\int f dS =$ Mass of S (think mass of a metal plate).

Example 4:

Calculate the mass of S, where S is the sphere of radius 3, with density $f(x, y, z) = z^2$

STEP 1: Picture:



STEP 2: Parametrize: Spherical with $\rho = 3$

$$r(\theta, \phi) = \langle 3\sin(\phi)\cos(\theta), 3\sin(\phi)\sin(\theta), 3\cos(\phi)\rangle$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \phi \leq \pi$$

STEP 3: Slopes and dS: Similar to the OMG Example above, can show:

$$dS = \|r_\theta \times r_\phi\| = 3^2\sin(\phi) = 9\sin(\phi)$$

STEP 4: Integrate

$$
\begin{aligned}
&\int\int_S z^2\, dS \\
=&\int_0^{\pi}\int_0^{2\pi} (3\cos(\phi))^2\, 9\sin(\phi) d\theta d\phi \\
=&2\pi\int_0^{\pi} 81\cos^2(\phi)\sin(\phi)d\phi \\
=&2\pi\left[\frac{-81}{3}(\cos(\phi))^3\right]_0^{\pi} \quad (\text{Use } u=\cos(\phi)) \\
=&2\pi(-27)(-1-1) \\
=&108\pi
\end{aligned}
$$

6. The case of Functions (again)

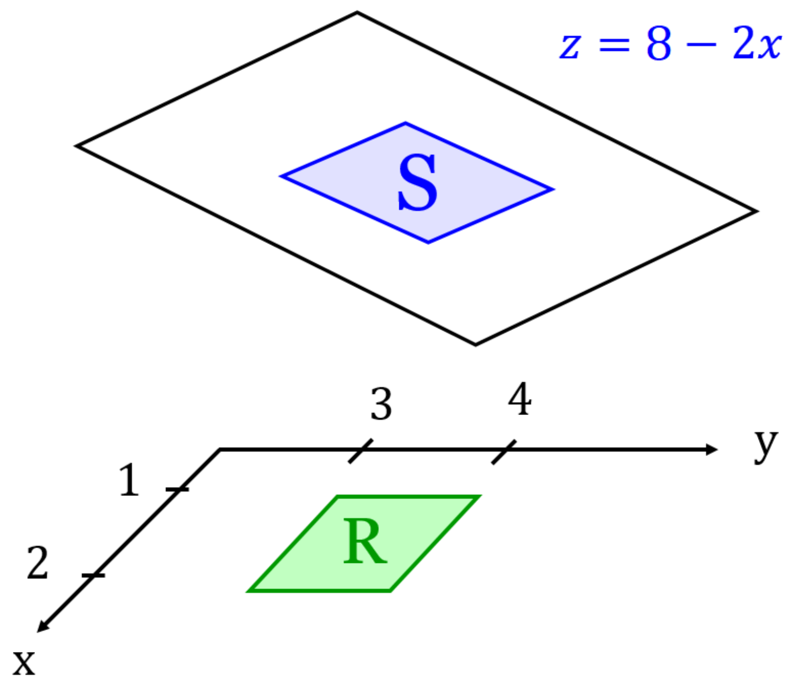
If S is the graph of a function, then the formula for $\int\int_S f dS$ simplifies:

Example 5: (extra practice)

$$\int\int_S x+2y+3z\, dS$$

S is the part of the plane $z=8-2x-y$ over the rectangle $R=[1,2]\times[3,4]$

STEP 1: Picture:



⚠ Here $z = 8 - 2x - y$ is our surface S, but $f(x, y, z) = x + 2y + 3z$ is the function we want to integrate. Do not confuse the two!

STEP 2: Parametrize

$$r(x, y) = \langle x, y, 8 - 2x - y \rangle$$

STEP 3: Slopes

$$r_x = \langle 1, 0, -2 \rangle$$
$$r_y = \langle 0, 1, -1 \rangle$$

STEP 4: Normal Vector

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = \langle 2, 1, 1 \rangle$$

STEP 5: dS

$$dS = \|r_x \times r_y\| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

STEP 6: Integrate:

$$\begin{aligned}
&\int\int_S x + 2y + 3z\, dS \\
=&\int\int_R (x + 2y + 3\,(8 - 2x - y))\,\sqrt{6}\,dxdy \\
=&\sqrt{6}\int_3^4 \int_1^2 x + 2y + 24 - 6x - 3y\, dxdy \\
=&\sqrt{6}\int_3^4 \int_1^2 -5x - y + 24\, dxdy \\
=&\sqrt{6}\int_3^4 \left[-\frac{5}{2}x^2 - yx + 24x\right]_{x=1}^{x=2} dy \\
=&\sqrt{6}\int_3^4 \frac{33}{2} - y\, dy \\
=&\sqrt{6}\left[\frac{33}{2}y - \frac{y^2}{2}\right]_3^4 \\
=&13\sqrt{6}
\end{aligned}$$

More Generally:

If $z = z(x, y)$ is the graph of a function, then

$$\int\int_S f(x, y, z)dS = \int\int_D f(x, y, z(x, y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

I would **not** memorize this formula if I were you.